Büchi Games for the Unguarded Alternation-free μ-Calculus

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Abstract

The modal μ-calculus is a fixpoint logic for the specification of ω-regular properties over labelled transition systems. It is known that alternation-free μ-calculus formulas – not having interdependent nesting of least and greatest fixpoints – generally correspond to co-Büchi automata. Existing satisfiability checking algorithms for *unguarded* μ-calculus formulas however rely on using full parity automata to detect activity of formulas along local evaluation cycles and hence do not exploit the correspondence to co-Büchi automata. Rather they incorporate full Safra-Piterman determinization for Büchi automata to reduce the satisfiabiliy problem to the solution of parity games. We propose an alternative construction that does *not* assume guardedness, yet reduces satisfiability of alternation-free μ-calculus formulas to Büchi games, sidelining the Safra-Piterman construction by using the simpler Miyano-Hayashi construction for co-Büchi automata instead.

1 Introduction

The modal μ-calculus is an expressive specification language that allows for expressing safety, reachability, and general fairness properties over transition systems [7]. Its main decision problems – model checking and satisfiability checking – are closely related [12, 10] to parity games, which have seen particular attention in recent research [1]. In the current work, we are interested in the satisfiability problem for (a fragment of) the μ-calculus. The standard procedure to solve this problem is by reduction to parity games [4, 6]. Crucially, the existence of least fixpoint formulas in the μ-calculus introduces the requirement that models must not contain infinite evaluation sequences for least fixpoints. Game reductions typically use non-deterministic ω-automata (*tracing automata*) to detect such sequences. The reduction to games then determinizes the tracing automaton and uses the determinized automaton as a game arena.

Alternation-free μ-calculus formulas do not have nested least and greatest fixpoints. For such formulas, the associated tracing automata are known to be co-Büchi automata (rather than parity or Büchi automata) which allow for simpler determinization [5]. However, this approach assumes that fixpoint variables in formulas are *guarded* by modal operators, ruling out infinite sequences of formula evaluations at a single state in the system. Existing methods that can deal with unguarded formulas rely on determinizing full parity automata via the more involved Safra-Piterman construction [11], and reduce satisfiability to parity games.

In the current work, we propose an algorithm that checks the satisfiability of alternation-free formulas, but does *not* assume guardedness. To this end, we adapt ideas from [3] to the alternation-free case, using co-Büchi tracing automata for global traces and treating local traces separately. This enables reduction to Büchi games via Miyano-Hayashi determinization [9], thereby enabling usage of the simpler methods for alternation-free formulas also without the assumption of guardedness. This improves the upper runtime bound on satisfability checking for the unguarded, alternation-free fragment. Furthermore, it shows that the correspondence betweeen alternation-free μ-calculus formulas and co-Büchi automata (and Büchi games, respectively) does not hinge on guardedness.

*Supported by the ERC Consolidator grant D-SynMA (No. 772459) and by the EPSRC through grant EP/Z003121/1

2 Automata, Games, and the μ-Calculus

We begin be recalling notions and results on ω-regular automata and infinite duration games.

An automaton is a tuple $\mathcal{A} = (\Sigma, Q, \delta, F)$, where Σ is a finite alphabet, Q is a finite set of states, $\delta : Q \times \Sigma \to 2^Q$ is a transition function, and $F \subseteq Q$ induces an *acceptance condition* $\alpha \subseteq Q^\omega$. In this work, we use Büchi and co-Büchi conditions; the former demand that some element of F is visited infinitely often, while the latter require that no element of F is visited infinitely often. If $|\delta(q, a)| \leq 1$ for all $q \in Q$ and $a \in \Sigma$, then we say that $\mathcal{A}$ is *deterministic* (and *non-deterministic* otherwise). A run of $\mathcal{A}$ on an infinite word is an infinite path through the automaton that is labelled with the word. A run of $\mathcal{A}$ on w is *accepting* if it is contained in α. The language *accepted* by $\mathcal{A}$ is $L(\mathcal{A}) = \{w \in \Sigma^\omega \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$.

Lemma 1 ([9])**.** *Let $\mathcal{A} = (\Sigma, Q, \delta, F)$ be a co-Büchi automaton. Then there is a deterministic co-Büchi automaton $\mathcal{A}'$ with at most $3^{|F|} \cdot 2^{|Q|-|F|}$ states such that $L(\mathcal{A}) = L(\mathcal{A}')$.*

A *Büchi game* is an infinite-duration game played by two antagonistic players $\exists$ and $\forall$, given as a tuple $G = (V, E, F)$, where $V = V_\exists \cup V_\forall$ is a set of nodes, partitioned into the sets $V_\exists$ of *existential nodes* and $V_\forall$ of *universal nodes*, $E \subseteq V \times V$ is a set of moves, and $F \subseteq V$ is a set of winning nodes. We put $E(v) = \{v' \in V \mid (v, v') \in E\}$. A *play* is a (finite or infinite) path through the graph (V, E). The existential player wins finite plays that end in $v_m \in V_\forall$ and infinite plays that visit some node from F infinitely often; all other plays are won by the universal player. The problem of *solving* a Büchi game consists in computing the set of game nodes for which the existential player has a strategy to win all plays starting at such a node.

Lemma 2 ([2])**.** *Büchi games with n nodes can be solved in time $\mathcal{O}(n^2)$.*

Next, we briefly introduce the modal μ-calculus, and its alternation-free fragment. Formulas of the μ-calculus are generated by the grammar

$$\varphi, \psi := p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi \mid X \mid \mu X.\, \varphi \mid \nu X.\, \varphi \qquad p \in \mathsf{At}, X \in \mathsf{Var}$$

where At and Var are countable sets of propositional atoms and fixpoint variables, respectively. Fixpoint operators give rise to standard notions of *free* and *bound* occurrences of fixpoint variables. We refer to fixpoint variables that are bound by a least fixpoint expression as *μ-variables* and to the remaining fixpoint variables as *ν-variables*.

Alternation-free formulas do not contain dependent nesting of least and greatest fixpoint expressions; formally, a formula is alternation-free if none of its subformulas contains free occurrences of both some μ-variable and some ν-variable. For instance the formula $\mu X.\, (\nu Y.\, p \wedge \Box Y) \vee \Diamond X$ is alternation-free while $\mu X.\, \nu Y.\, ((p \wedge \Box Y) \vee \Diamond X)$ is not alternation-free.

Guardedness of fixpoint variables requires that any free occurrence of a fixpoint variable X within a fixpoint expression $\eta X.\, \varphi$ is in the scope of at least one modal operator ($\Diamond$ or $\Box$). In this work, we do *not* assume guardedness of formulas, that is, the satisfiability algorithm presented below works for unguarded formulas such as for instance $\mu X.\, \mu Y.\, (X \vee \Diamond(Y \wedge p))$.

We define the *(Fisher-Ladner) closure* $\mathsf{FL}(\varphi)$ of a closed formula φ to be the least set of formulas that contains φ and is closed under taking subformulas and under *fixpoint unfolding* (which transforms formulas of shape $\eta X.\, \varphi$ to $\varphi[X \mapsto \eta X.\, \varphi]$, where $\eta \in \{\nu, \mu\}$). The *size* $|\varphi|$ of a formula φ is the size $|\mathsf{FL}(\varphi)|$ of its closure.

More details on these syntactic notions can be found, e.g., in [8].

Formulas are evaluated over Kripke structures in the standard way (cf. [7]) and a formula is *satisfiable* if it has a model, that is, if there is a Kripke structure that satisfies the formula.

3 Satisfiability of Unguarded Alternation-free Formulas

We fix an alternation-free but not necessarily guarded formula χ and denote its closure by FL.

We first define the tracing automaton for χ and then use it to devise an algorithm that checks whether there is a Kripke structure on which there is no μ-trace of χ, that is, no trace of χ that infinitely often unfolds a least fixpoint.

Definition 3 (Tracing automaton)**.** A *local strategy* is a function s that assigns a choice $s(\psi_0 \vee \psi_1) \in \{0,1\}$ to each disjunction $\psi_0 \vee \psi_1 \in \mathsf{FL}$. We let loc denote the set of all local strategies and put $\Sigma = \mathsf{loc} \cup \{\Diamond\varphi \mid \Diamond\varphi \in \mathsf{FL}\}$. The *tracing automaton* for χ is the nondeterministic co-Büchi automaton $\mathcal{A}_\chi = (\Sigma, \mathsf{FL}, \delta, F)$ defined by putting $F = \{\nu X.\, \varphi \mid \nu X.\, \varphi \in \mathsf{FL}\}$. The transition function is defined, for $\varphi \in \mathsf{FL}$ and $s \in \mathsf{loc} \subseteq \Sigma$, by

$$\delta(\psi_0 \wedge \psi_1, s) = \{\psi_0, \psi_1\} \qquad \delta(\psi_0 \vee \psi_1, s) = \{\psi_{s(\psi_0 \vee \psi_1)}\} \qquad \delta(\eta X.\, \psi, s) = \{\psi[\eta X.\, \psi / X]\}$$
$$\delta(\Diamond\psi, \Diamond\psi) = \{\psi\} \qquad \delta(\Box\varphi, \Diamond\psi) = \{\varphi\}$$

In all other cases, we put $\delta(\psi, s) = \{\psi\}$ and $\delta(\varphi, \Diamond\psi) = \emptyset$. Therefore, the only non-deterministic transitions in δ arise at formulas of the shape $\varphi_1 \wedge \varphi_2$ when reading a letter $s \in \mathsf{loc}$.

Let $\mathcal{D}_\chi = (\Sigma, S, \Delta, B)$ be the determinized version of $\mathcal{A}_\chi$ with $|S| \leq 3^{|\varphi|}$ obtained by Lemma 1. States $q \in S$ are of the shape $q = (U, W)$ for $W \subseteq U \subseteq \mathsf{FL}$. We write $\varphi \in q$ if $\varphi \in U$.

A local strategy s is *admissible* for a set $U \subseteq \mathsf{FL}$ of formulas, if there is no formula $\mu X.\, \psi \in \mathsf{FL}$ such that s reproduces $\mu X.\, \psi$, starting from U. Intuitively, a local strategy is admissible for U if it does not determine a local cycle (not involving any modal steps) in δ that contains a least fixpoint formula. Given $q = (U, W) \in S$, we let $H(q)$ denote the set of all admissible local strategies for U.

Next we reduce the satisfiability check for χ to the solution of a Büchi game played over $\mathcal{D}_\chi$. The existential player provides *admissible local strategies* for a state while the universal player picks one existential modal formula from a given state and applies the according modal step. A play is winning for the existential player if and only if the according run of $\mathcal{D}_\chi$ is *not* accepting, that is, does not contain a μ-trace.

Definition 4. The *satisfiability game* for χ is the Büchi game $G_\chi = (S \times \{0,1\}, E, (S \setminus B) \times \{0,1\})$, having $2 \cdot |S| \leq 2 \cdot 3^{|\varphi|}$ nodes. The following table completes the definition of G_χ.

node	owner	moves to
$(q,0)$	$\exists$	$\{(\Delta(q, s^*), 1) \mid s \in H(q)\}$
$(q,1)$	$\forall$	$\{(\Delta(q, \Diamond\varphi), 0) \mid \Diamond\varphi \in q\}$

The game G_χ alternates between propositional stages $(q, 0)$ and modal stages $(q, 1)$. In propositional stages, player $\exists$ picks a local strategy s that is admissible for q and repeatedly applies this strategy to q (denoted by s^*), resulting in a move to the modal stage $(\Delta(q, s^*), 1)$. In a modal stage, player $\forall$ picks a diamond formula from q and moves on to $(\Delta(q, \Diamond\varphi), 0)$.

Lemma 5. *The existential player wins G_χ if and only if χ is satisfiable.*

The proof of Lemma 5 constructs models over S, showing that satisfiable alternation-free (but not necessarily guarded) formulas φ have models of size at most $3^{|\varphi|}$. We sum up the results of the current work as follows.

Corollary 6. *The satisfiability of alternation-free μ-calculus formulas φ can be checked in time $\mathcal{O}(3^{2|\varphi|})$. If φ is satisfiable, then it has a model of size at most $3^{|\varphi|}$.*

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