REGULAR HEYTING ALGEBRAS AND FREE HEYTING EXTENSIONS OF BOOLEAN ALGEBRAS

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1. Free Heyting Extensions

The categories **HA** and **BA**, respectively of Heyting algebras with Heyting algebra homomorphisms and Boolean algebras with Boolean algebra homomorphisms, are related by a chain of adjunctions:

$$\mathsf{FreeM} \dashv \mathsf{Reg} \dashv I \dashv \mathsf{Center},$$

where

(1) $I : \mathbf{BA} \to \mathbf{HA}$ is the inclusion;
(2) $\mathsf{Center} : \mathbf{HA} \to \mathbf{BA}$ takes the *center* of a Heyting algebra H, namely $\mathsf{Center}(H) = \{a \in H : a \vee \neg a = 1\}$;
(3) $\mathsf{Reg} : \mathbf{HA} \to \mathbf{BA}$ takes the *regular elements* of a Heyting algebra H namely $\mathsf{Reg}(H) = \{a \in H : \neg\neg a = a\}$.

The functor $\mathsf{FreeM} : \mathbf{BA} \to \mathbf{HA}$ is guaranteed to exist by the Adjoint Functor theorem; a description using a word construction can be derived from the work of Moraschini [6] (see also [7, Example 4.18]). It was likewise studied in [8], where it was shown to be fully faithful.

In this paper we give a more concrete presentation of the functor FreeM, by studying the dual problem, employing Esakia duality (see [4] for any unexplained notions such as Priestley spaces, Esakia spaces and p-morphisms): the dual to Reg is the functor $\mathsf{Max} : \mathbf{Esa} \to \mathbf{Stone}$, mapping an Esakia space $(X, \leq)$ to its maximum. Here we present the right adjoint to this functor, also denoted by FreeM, by borrowing key ideas from [1]. Essentially, this amounts to finding some free Heyting extension of a Boolean algebra which still has the same regular elements.

Given X a Stone space, let $(V(X), \supseteq)$ be the Priestley space of its closed subsets. The following is presumably folklore:

Proposition 1. *For each Stone space X, $(V(X), \supseteq)$ is an Esakia space.*

Our goal will be, given X an Esakia space, and Y a Stone space, to extend a function $f : \max X \to Y$ to a unique p-morphism $\tilde{f} : X \to \mathsf{FreeM}(Y)$. The first step is to note that $V(X)$ enjoys a universal property related to this:

Proposition 2. *Assume that X is an Esakia space, Y is a Stone space and $f : \max X \to Y$ is a continuous map. Then there exists a unique order-preserving map $\tilde{f} : X \to V(Y)$, such that $e_Y \circ \tilde{f} \restriction_{\max} = f$ and $\tilde{f}$ is a p-morphism on maximal elements (i.e., if $\tilde{f}(x) \leq y$ and $y \in V(Y)$ is maximal, then there is some $w \geq x$ such that $\tilde{f}(w) = y$).*

Relatedly, one can consider a variation of the rooted Vietoris construction from [1]:

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Definition 3. Given two Priestley spaces X, Y and a continuous and order-preserving map $g : X \to Y$ between them, we say that a subset $S \subseteq X$ is *g-open* if it satisfies:

$$\forall x \in S, y \in X(x \leq y \to \exists z \in S(x \leq z \wedge g(z) = g(y))).$$

We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

Note that if $Y = \{\bullet\}$, and g is the terminal map, $V_g(X)$ is the set of all closed and rooted subsets, which we denote by $V_r(X)$. Recall that there is a map called the *root map* $r : V_g(X) \to X$ which is a surjective order preserving map.

Proposition 4. *Let Y be a Stone space and let $V_{\max}(Y) = \{C \in V_r(V(Y)) : \forall D \in C, \forall x \in D, \{x\} \in C\}$. Then $V_{\max}(Y)$ is a Priestley space, and the restriction $r : V_{\max}(Y) \to V(Y)$ is such that for any map $f : \max X \to Y$, and its unique lifting $\tilde{f} : X \to V(Y)$ from Proposition 2, there is a unique r-open $g_f : X \to V_{\max}(Y)$ making the diagram commute.*

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} ...$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, as in [1], and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V_G^r(V_{\max}(Y))$ is the inverse limit of this sequence. $M_\infty(Y)$ is then an Esakia space, with the property that $\max(M_\infty(Y)) \cong Y$ through a natural isomorphism; moreover this assignment is functorial by using the functoriality of $V(-)$, $V_{\max}(-)$ and $V_G^r(-)$.

Our main result then shows the following adjunction:

Proposition 5. *The functor* $\mathsf{FreeM} : \mathbf{Stone} \to \mathbf{Esa}$ *assigning each Stone space X to $M_\infty(X)$ is right adjoint to* $\max : \mathbf{Esa} \to \mathbf{Stone}$.

2. Regular Heyting Algebras

We use the description of FreeM to study *regular Heyting algebras*, and their dual, regular Esakia spaces, in the sense of [5].

Definition 6. Let H be a Heyting algebra. We say that H is *regular* if $H = \langle \mathsf{Reg}(H) \rangle$. We say that an Esakia space X is regular, if its dual Heyting algebra is regular.

These are known to provide algebraic (respectively, order-topological) semantics for inquisitive logic [2]; indeed, given B a Boolean algebra, the Heyting algebra $\mathsf{ClopUp}(V(X_B))$ is studied there as the *inquisitive extension* of B. Using our construction, we show:

Theorem 7. *Given a Stone space X, $M_\infty(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are the algebras for the monad induced by this functor.*

This result can be seen as a dual algebraic semantics for inquisitive logic, and provides a categorical explanation for the role played by regular Heyting algebras in the study of such a logic. This furthermore answers a question of Grilleti and Quadrellaro[5], providing a categorical description of regular Esakia spaces. As an elaboration on the main ideas used to prove this theorem, we provide two theorems concerning regular Heyting algebras.

Definition 8. Given $n \in \omega$ the *n-universal regular model* if the (unique) poset $(\mathcal{R}_n, \leq)$ satisfying the following:

(1) $\max(P)$ contains 2^n points.

(2) For each antichain $S \subseteq R_n$ where $|S| \geq 1$, there is a unique point $x \in P$ which covers S.

Compared with the usual n-universal model, the regular such model can be obtained by identifying all points with the same color, and hence it is a p-morphic image of the n-universal model. Then we can show:

Theorem 9. *Inquisitive logic* InqL *is sound and complete with respect to the class* $\{\mathcal{R}_n : n \in \omega\}$.

We furthermore introduce the following notion:

Definition 10. Let $n \in \omega$. We say that a Heyting algebra H is n-regular if H is generated from $\mathsf{Reg}(H)$ by formulas of implication depth at most n.

Then we can show[1]:

Theorem 11. *If H is a Heyting algebra, then H is 0-regular if and only if H is a homomorphic image of an algebra* $\mathsf{ClopUp}(V(X))$ *for X a Stone space. In the finite case, H is a homomorphic image of the dual of* $M_n \cong \mathcal{P}(n) - \{\emptyset\}$.

3. Connections with Medvedev's Logic

We conclude by tracing some connections and further avenues of research connected with the intermediate logic ML, Medvedev's logic. This is well known to be the logic of the posets $V(X)$ for X a finite Stone space, i.e., a finite set, since $V(X)$ is exactly the powerset without the empty set. It likewise appears naturally in the study of inquisitive logic, as the *schematic fragment* of InqL. Here we prove:

Theorem 12. *The logic* ML *is precisely the logic of all the spaces $V(X)$ for X a Stone space, and hence, the logic of all 0-regular Heyting algebras.*

In light of this result, and the fact that IPC is the logic of all regular Heyting algebras [3], we introduce the following hierarchy:

Definition 13. We define $\mathsf{R}_n := \mathsf{Log}(\{H : H \text{ is } n\text{-regular}\})$.

We will present some preliminary results concerning this hierarchy and discuss how it connects to other approaches in the study of ML.

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