Relevant Epistemic Logic and Knowledge as Belief Based on Correct Evidence

Igor Sedlár*

Czech Academy of Sciences, Institute of Computer Science
Prague, Czech Republic

1 Introduction

Relevant logics avoid some problematic principles of classical and intuitionistic logic, such as explosion – everything follows from a contradiction – or the paradoxes of implication – for example, that every implication follows from its consequent [7,12]. Relevant epistemic logics are epistemic logics based on relevant logics [1,3,8,13,14,15,16,17,19,20,21,22]. The motivation for studying them is that they provide a much finer-grained model of epistemic attitudes than that provided by classical epistemic logic. Relevant epistemic logics avoid many problematic closure principles rooted in classical logic – for example, a belief base may be inconsistent but non-trivial, or it may contain a consequent of an implication without containing the implication itself. In addition, informational interpretations of relevant logics [6,7,9,11] suggest that relevant logics provide a natural framework for epistemic logic.

Existing relevant epistemic logics model variants of *evidence-based belief.* In this paper we focus on the interplay between evidence, belief and *knowledge.* This fills a gap in the literature and complements recent extensions of relevant logics with knowledge-like operators [22]. In particular, we present a reductionist framework in which belief and knowledge are based on various kinds of evidence. Our main technical result is a representation theorem for a certain class of distributive lattices with operators. This theorem entails a completeness result for a large family of reductionist relevant epistemic logics of knowledge, evidence and belief.

2 Semantics

For the sake of exposition, we will first discuss the core of our approach in the context of simplified semantic structures that lack much of the complexity inherent in the relational semantics of relevant logics. We comment on the significance of this simplified setup to full relevant logics in Section 4.

Definition 1. *A* proto-frame *is a partially ordered set* $\langle S, \leqslant \rangle$ *together with two binary relations* E_r *and* E_c *such that, for* $E \in \{E_r, E_c\}$*, if* Est *and* $t \leqslant u$*, then* Esu*.*

Proto-frames represent a set of *situations* (representations of the world which may be partial or inconsistent; see [7]), partially ordered by the amount of information they support. Assuming a fixed agent, the relation E_r represents *evidence recognised by the agent* – $E_r st$ means that situation t contains information that the agent recognises the situation s as evidence on the basis of which the agent forms their epistemic attitudes.[1] The relation E_c represents *correct evidence* – $E_c st$ means that s is correct evidence given the information in t. We do not aim at arguing for a specific analysis of what correctness of evidence consists in – this is a long-standing epistemological problem – we aim only at formulating a mathematical

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[1] The monotonicity condition $E_r st \;\;\&\;\; t \leqslant u \implies E_r su$ is "objective" – it means that adding information to t does not negate the information that the agent recognises s, not that the agent will recognise s no matter what new information the agent comes to accept.

framework in which various analyses can be represented. The monotonicity condition $E_c st$ & $t \leqslant u \implies E_c su$ means that correct evidence is *indefeasible* in the sense that adding information to t will not negate the information that s is correct.[2]

Definition 2. *Given a proto-frame* $\langle S, \leqslant, E \rangle$*, a* proposition *is a set of situations closed upwards under the order* $\leqslant$ *of the proto-frame. We define two operations on propositions:*

$$B(X) = \{t \mid \exists s(E_r st \ \& \ s \in X)\} \qquad K(X) = \{t \mid t \in X \ \& \ \exists s(E_r \cap E_c)st \ \& \ s \in X)\}$$

The monotonicity frame condition in Definition 1 ensures that B and K are indeed operations on propositions, that is, if X is a proposition, then $B(X)$ and $K(X)$ are propositions.

We say that $t \in S$ *supports* a proposition X if $t \in X$. Hence, propositions need to be closed under the partial order – if s supports X, then so does every situation that is at least as strong as s. Now t supports $B(X)$ if, according to t, the agent recognises some evidence that supports X. We claim that B is a natural and simple representation of evidence-based *belief*. A similar operator was used to represent evidence-based belief in [1]. On the other hand, t supports $K(X)$ if t supports X and there is *correct* evidence recognised by the agent that supports X. We claim that K is a natural *knowledge* operator. Invoking the intersection of E_r and E_c is crucial, as witnessed by Gettier-type cases [4] where knowledge is not obtained since the agent believes a true proposition based on incorrect evidence.

Definition 3. *A* proto-algebra *is a distributive lattice with unary operations* B *and* K *s.t.*

$$Kx \leq Bx \wedge x \qquad B(x \vee y) = Bx \vee By \qquad K(x \vee y) = Kx \vee Ky$$

A natural example of a proto-algebra is the proto-algebra *based on a proto-frame*, that is, the collection of propositions on the proto-frame with intersection, union, and B and K as defined in Definition 2. While proto-frames are semantic structures based on situations taken as primitive, proto-algebras are based on propositions taken as primitive. The interplay between proto-frames and proto-algebras, clarified in Section 3, is crucial for establishing completeness for relevant epistemic logics, as discussed in Section 4.

3 A representation result

Theorem 1. *Every proto-algebra* A *embeds to a proto-algebra* A' *based on a proto-frame.*

Proof (sketch). Take A and define the following proto-frame, called the *proto-frame of* A. Firstly, S is the collection of prime filters on A ordered by set inclusion. Secondly, we define a binary relation E_K on S such that $E_K st$ iff $x \in s$ implies $Kx \in t$ for all $x \in A$. Note that $E_K st$ implies $s \subseteq t$ thanks to $Kx \leq x$. Thirdly, we define $E_r st$ iff $Bx \in t$ for all $x \in s$. Fourthly, we define $E_c st$ iff $E_K st$ or, for all $u \geqslant t$, not $E_r su$. Note that $E_K \subseteq E_r$ thanks to $Kx \leq Bx$ and so $E_r \cap E_c = E_K$. Note also that both E_r and E_c are monotone in their second position and so this structure is indeed a proto-frame. A standard argument shows that $Bx \in t$ iff there is s such that $E_r st$ and $x \in s$. A similar argument shows that $Kx \in t$ iff $x \in t$ and there is s such that $x \in s$ and $E_K st$.[3] Let A' be the proto-algebra based on our proto-frame. We define a mapping e from A to A' such that $e(x)$ is the set of prime filters in S that contain x. The previously stated facts show that e is a homomorphism. It is also injective by well-known properties of distributive lattices; see [5], p. 84, for example. □

[2] Note that we do not require $E_c st \implies s \leqslant t$. Situation s may be correct evidence given t, but t itself does not need to support "first hand" all information supported by s.

[3] Right to left: If $x \in s$ and $E_K st$, then $Kx \in t$ by the definition of E_K. Left to right: Assume that $Kx \in t$. Take the pair $\langle \{x\}, \{z \mid Kz \notin t\} = Z \rangle$. This pair is *independent* in the sense that $x \not\leq \bigvee Z'$ for all non-empty $Z' \subseteq Z$. Z is a (possibly empty) ideal disjoint from the filter $X = \{x' \mid x' \geq x\}$ and so there is a prime filter s extending X that is disjoint from Z. (If Z is empty then we can take $s = A$.) Clearly $x \in s$ and $E_K st$. Moreover, $Kx \in t$ implies $x \in t$ using $Kx \leq x$.

4 Discussion

Standard arguments (see [2], [10]) show that Theorem 1 can be used to establish a completeness result for any distributive relevant logic with B and K accompanied by the obvious axioms derived from Definition 1, with respect to a suitable relational semantics. This semantics is established by extending proto-frames to *frames* by adding the usual relevant machinery: a set of normal worlds N, a ternary accessibility relation R for implication and fusion (intensional conjunction) and either the Routley star * or a compatibility relation C for negation.[4] We omit the details.

Relatively standard arguments can also be used to show that specific frame conditions correspond to specific equations defining varieties of proto-algebras in the sense that (i) if the equation is valid in a proto-algebra, then the proto-frame of the proto-algebra satisfies the frame condition, and (ii) if a proto-frame satisfies the frame condition, then the equation is valid in the proto-algebra based on the proto-frame. For example, Stalnaker's [18] axioms of *strong belief* $Bx \leq BKx$ and *positive introspection* $Bx \leq KBx$ correspond to

(SB) $E_r st \Rightarrow \exists u((E_r \cap E_c)su \,\&\, s \leqslant u \,\&\, E_r ut)$ (PI) $E_r st \Rightarrow \exists u(E_r su \,\&\, (E_r \cap E_c)ut)$

respectively.[5] These observations lead to completeness results for stronger logics that adopt some non-trivial assumptions concerning knowledge and belief.

A feature of many evidence-based relevant epistemic logics, e.g. [1], is that evidence is modelled as being *prime* – if there is evidence for $X \vee Y$, then there is evidence for X or there is evidence for Y. Our basic framework shares this feature, but it can be generalised to avoid it. One way to do this is to use neighbourhood functions instead of relations E and E_C. We discuss such generalisations in the full version of the paper.

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[4] In fact, our representation result leads to completeness results for many intuitionistic and intermediate epistemic logics as well. In the relational semantics for these logics, one does not need to add extra relations to deal with implication and negation as these are interpreted using the partial order.

[5] (i) To show that the proto-frame of A satisfies (SB) if $Bx \leq BKx$ for all $x \in A$, assume that $E_r st$ and we will construct an appropriate u. Take the pair $\langle \{Kx \mid x \in s\}, \{y \mid By \notin t\}\rangle$. It can be shown using $Bx \leq BKx$ and properties of proto-algebras that the pair is independent (see footnote 3), and so there is u that contains $\{Kx \mid x \in s\}$ and is disjoint from $\{y \mid By \notin t\}$. Obviously $E_r ut$ and $(E_r \cap E_c)su$. (The latter is based on the fact that we are using the definition of a proto-frame of A from the proof of Theorem 1, where it turned out that $(E_c \cap E_r)su$ iff $x \in s$ implies $Kx \in u$.) Moreover, if $x \in s$, then $Kx \in u$, and so $x \in u$ thanks to $Kx \leq x$, which means that $s \subseteq u$. (ii) is straightforward.

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