A Deep-Inference Sequent Calculus for a Propositional Team Logic

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Abstract

Logics such as *dependence logic* [22] and *inquisitive logic* [8] are usually interpreted using *team semantics* [13, 14]: formulas are evaluated with respect to sets of evaluation points (valuations/assignments/possible worlds) called *teams*, rather than single evaluation points as in the usual Tarskian semantics. In the propositional setting we focus on, teams are sets of propositional valuations. Team semantics allows for the introduction of non-classical connectives such as the *inquisitive* or *global disjunction* $\vee\!\!\!\vee$:

for a team t: $\quad t \models \phi \vee\!\!\!\vee \psi \iff t \models \phi \text{ or } t \models \psi;$

cf. the *split disjunction* $\vee$: $\quad t \models \phi \vee \psi \iff \exists s, u : t = s \cup u, s \models \phi, \text{ and } u \models \psi,$

where it is the split rather than the inquisitive disjunction which generalizes the usual disjunction of single valuation-based semantics in the team-based setting. These team-semantic connectives provide for a natural way to formalize notions such as question meaning ('p or not p?' can be formalized as $p \vee\!\!\!\vee \neg p$, as done in inquisitive logic) and dependence ('the value of q functionally depends on the value of p' can be formalized as $(p \wedge (q \vee\!\!\!\vee \neg q)) \vee (\neg p \wedge (q \vee\!\!\!\vee \neg q))$, as essentially done in dependence logic).

While there are a great number of natural deduction- and Hilbert-style axiomatizations of propositional team logics in the literature [8, 19, 17, 25, 6, 26, 15, 7, 23], the development of sequent calculus systems and of proof theory in general for these logics has been slower. The sequent calculi that have been constructed have all been for variants of propositional inquisitive logic; these include multiple labelled systems [5, 16, 1] as well as the multi-type display calculus in [10].

One of the main difficulties in providing standard sequent calculi for team logics is that usually these logics are not *closed under uniform substitution.* For instance, the entailment $p \vee p \models p$ holds, but substitution instances of the entailment might not: $(p \vee\!\!\!\vee \neg p) \vee (p \vee\!\!\!\vee \neg p) \not\models (p \vee\!\!\!\vee \neg p)$. Due to this failure, axiomatizations for team logics typically feature rules that may only be applied to some subclass of formulas, and these axiomatizations do not admit the usual uniform substitution rule. Many proof-theoretic techniques depend on the universal applicability of the rules, so it is not immediately obvious how to apply these techniques to most team logics—often some specialized machinery has to be introduced to handle this issue. For instance, the construction of the multi-type display calculus in [10] involves the introduction of a new language featuring two types of formulas for the team logic axiomatized, with closure under substitution holding within each of these types. It is also not even immediate how a sequent should be interpreted in the setting of team semantics—for instance, one may interpret the commas in the succedent Δ of a sequent $\Gamma \Rightarrow \Delta$ using the split disjunction $\vee$ or the inquisitive disjunction $\vee\!\!\!\vee$.

We propose a novel way of addressing these difficulties by way of introducing a sequent calculus featuring some *deep-inference*-style rules (see, e.g., [20, 18, 11, 4]) for the propositional team logic $\mathbf{CPL}(\vee\!\!\!\vee)$ with both the split disjunction $\vee$ and the inquisitive disjunction

$\mathbin{\vee\!\!\!\!\vee}$ (this logic, essentially a modest extension of propositional dependence logic, is studied in, e.g., [25, 19, 6, 24]). Our calculus consists of a standard Gentzen-style system (a variant of G3cp [21]) for the fragment of the logic which corresponds to classical propositional logic **CPL** (the $\mathbin{\vee\!\!\!\!\vee}$-free fragment), with some syntactic restrictions to the effect that certain active formulas and context sets must be classical ($\mathbin{\vee\!\!\!\!\vee}$-free); together with deep-inference-style rules for the inquisitive disjunction $\mathbin{\vee\!\!\!\!\vee}$—that is, rules which allow one to introduce the inquisitive disjunction (almost) anywhere within a formula, rather than only as its main connective:

$$\frac{\Gamma, \chi\{\phi_1\} \Rightarrow \Delta \qquad \Gamma, \chi\{\phi_2\} \Rightarrow \Delta}{\Gamma, \chi\{\phi_1 \mathbin{\vee\!\!\!\!\vee} \phi_2\} \Rightarrow \Delta}\, L\mathbin{\vee\!\!\!\!\vee} \qquad \frac{\Gamma \Rightarrow \chi\{\phi_i\}, \Delta}{\Gamma \Rightarrow \chi\{\phi_1 \mathbin{\vee\!\!\!\!\vee} \phi_2\}, \Delta}\, R\mathbin{\vee\!\!\!\!\vee}$$

(Here ϕ_1 is a subformula occurrence of the formula $\chi\{\phi_1\}$ that does not occur in the scope of a negation, $\chi\{\phi_2\}$ is $\chi\{\phi_1\}$ with ϕ_2 replacing ϕ_1, and similarly for $\chi\{\phi_1 \mathbin{\vee\!\!\!\!\vee} \phi_2\}$.)

The deep-inference-style rules allow for cutfree completeness of the system (cut is admissible in the cutfree fragment of the system) and for many standard proof-theoretic techniques to be applied despite the limited applicability of the restricted rules: essentially, cutfree proofs can be constructed by first constructing cutfree classical proofs, and then introducing inquisitive disjunctions as required; and procedures involving the permuting of sequents which depend on the universal applicability of the rules (such as cut elimination) can be conducted in such a way that they only involve permuting sequents in the classical part of the calculus, in which the rules *are* universally applicable.

The resulting system departs minimally from a Gentzen-style calculus. Our approach avoids both importing the semantics into the system in the form of labels (like the labelled systems in [5, 16, 1]) and extending the syntax of the logic (like the multi-type display calculus in [10]). Our system also consists of only a single pair of rules for each connective—this is in contrast with the frequently complex natural deduction systems for team logics, including the natural deduction system for the logic we focus on [25].

Furthermore, given that the system is a natural extension of a well-known Gentzen-style system for **CPL** with rules for the inquisitive disjunction $\mathbin{\vee\!\!\!\!\vee}$, the fact that **CPL**($\mathbin{\vee\!\!\!\!\vee}$) is an extension of **CPL** with $\mathbin{\vee\!\!\!\!\vee}$ is directly reflected in a straightforward way in the calculus, and the calculus allows us to see immediately and transparently exactly what is required to be added to an axiomatization of **CPL** to axiomatize **CPL**($\mathbin{\vee\!\!\!\!\vee}$). This structure is possible due to a key design decision: instead of interpreting a sequent $\Gamma \Rightarrow \Delta$ as $\bigwedge \Gamma \models \mathbin{\vee\!\!\!\!\vee}\Delta$ (i.e., taking the sequent $\Gamma \Rightarrow \Delta$ to be valid just in case whenever each formula in Γ is true in a team t, at least one formula of Δ is true in t), we interpret $\Gamma \Rightarrow \Delta$ as $\bigwedge \Gamma \models \bigvee \Delta$ (i.e., $\Gamma \Rightarrow \Delta$ is valid just in case whenever each formula in Γ is true in a team t, there is, for each $\phi \in \Delta$, a t_ϕ such that $t_\phi \models \phi$; and $t = \bigcup_{\phi \in \Delta} t_\phi$). That is, we interpret the comma in the succedent of a sequent not as the inquisitive disjunction $\mathbin{\vee\!\!\!\!\vee}$ (as is done in the labelled systems in [5, 16, 1]), but, rather, as the split disjunction $\vee$.

Given that our calculus extends G3cp in a straightforward manner, we are able to show many proof-theoretic results for our system as extensions or corollaries of the analogous results for G3cp. These results include the height-preserving admissibility (with some restrictions) of weakening, contraction and inversion; a G3cp-style proof of cutfree completeness and countermodel construction; a cut-elimination procedure that relies on a normal form theorem for derivations to the effect that any cutfree derivation can be converted into one in which one first applies only G3cp-rules, following which one only applies the $\mathbin{\vee\!\!\!\!\vee}$-rules (cf. the *decomposition theorems* in, e.g., [11, 12, 2]); and a sequent interpolation theorem.

The system also allows us to prove—via the interpolation theorem—some novel results concerning the logic **CPL**($\mathbin{\vee\!\!\!\!\vee}$). The literature on interpolation in propositional team logics [9, 7, 23] has mainly focused on uniform interpolation—there are (as far as we know) no results on constructing Craig's interpolants that are not also uniform interpolants, and no results on Lyndon's interpolation. Our sequent interpolation theorem yields as corollaries both a Lyndon's interpolation theorem as well as a Craig's interpolation theorem that does not rely on the construction of (the comparatively more complex) uniform interpolants.

One further interesting feature of the system is that due to the decision to interpret the succedent comma as the split disjunction, there is a correspondence between certain structural rules of the calculus and certain *team-semantic closure properties*—semantic properties of formulas which play an important role in the study of team logics. The closure properties which have structural correspondents in our system are the *empty team property* and *union closure*. The logic $\mathbf{CPL}(\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}})$ has the empty team property (meaning that the empty team satisfies all $\mathbf{CPL}(\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}})$-formulas), and its classical fragment is union closed (meaning that the truth of a classical formula in a collection of teams implies its truth in the union of the collection); $\mathbf{CPL}(\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}})$ as a whole is not union closed. The empty team property corresponds to the soundness of weakening on the right, and union closure corresponds to the soundness of contraction on the right; accordingly, weakening on the right is sound (and admissible in the cutfree fragment of our system) for all $\mathbf{CPL}(\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}})$-formulas, whereas contraction on the right is only guaranteed to be sound (and admissible in the cutfree fragment) for classical formulas.

The formulation of our system as an extension of $\mathsf{G3cp}$ minimizes the amount of proof-theoretic machinery required, but we also provide a similar system for $\mathbf{CPL}(\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}})$ in the *calculus of structures* [11], a deep-inference formalism that allows for *all* rules to be applied at any depth in a formula. This alternative system in the calculus of structures is an extension of the system SKSg [3] for classical propositional logic with structures that translate the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, and with rules governing the behaviour of these structures.

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