A Unified Approach to the Analysis of the Expressiveness of Team Logics

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Team semantics is a generalization of Tarski semantics in which logical formulae are not evaluated for single assignments, but for *sets of assignments* called *teams*. This opens new avenues to reason about interdependencies between assignments, which are relevant e.g. for large sets of data. In fact, most prominent team logics feature notions that have been studied in database theory like dependence [3], independence [11], inclusion [4] or exclusion [5]. Other applications include linguistics [7] or quantum logics [1, 16].

Team semantics was originally conceived by Hodges [14] to provide a compositional model-theoretic semantics for independence-friendly logic [13]. Since then, it has been established as a basis for logics of imperfect information. Here, it is prevalent to augment basic first-order team logic (FOT) with *team atoms* that correspond to interdependencies between assignments. This approach emerged with Väänänen's dependence logic FOT(dep) [22] and includes (conditional) independence logic [12] FOT(indep) and inclusion/exclusion logic FOT(inc, exc) [8].

A significant effort has been made to analyse the expressive power of these logics. Mostly, this is done by comparison to (fragments of) existential second order logic (ESO) in the sense that we find a correspondence between a set of team atoms $*$ and a semantic property P of ESO(R)-sentences:

Theorem (a typical classification result)**.** *For all* $\varphi(R) \in \mathrm{ESO}$ *with property* P *there is a* $\varphi^{\#} \in \mathrm{FOT}(*)$ *(and vice versa) such that*

$$\mathfrak{A}[T(\bar{y})/R] \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{A}, T \models \varphi^{\#}$$

for all structures $\mathfrak{A}$ *and teams* T *with* $\mathrm{dom}(T) = \bar{y}$.[1]

An example for such a semantic property is downward closure: a formula $\varphi(R) \in \mathrm{ESO}$ is downward closed if for all interpretations $R_1^{\mathfrak{A}} \subseteq R_2^{\mathfrak{A}}$ of R, we have $\mathfrak{A}\left[R_2^{\mathfrak{A}}/R\right] \models \varphi$ implies $\mathfrak{A}\left[R_1^{\mathfrak{A}}/R\right] \models \varphi$. Dependence logic is equivalent to the downward closed fragment of ESO [17]. Further, independence logic and inclusion/exclusion logic are both equivalent to full ESO [8]. Inclusion logic is union closed, but only equivalent to the so-called myopic fragment of $\nu\mathrm{FO}^+$ [9]. The union closed fragment of ESO has been established as equivalent to FOT with "$\cup$-game"-atoms that correspond to winning strategies in specific model checking games [15]. For sentences, dependence, independence and exclusion logic are all equivalent to ESO [22, 8] and inclusion logic is equivalent to positive greatest fixpoint logic $\nu\mathrm{FO}^+$ [9].

The main contribution of the paper consist of two parts.

[1]Here, $T(\bar{y}) = \{(t(y_1), \ldots, t(y_n)) \mid t \in T, \bar{y} = (y_1, \ldots, y_n)\}$ is the relational encoding of T and $\mathfrak{A}[T(\bar{y})/R]$ is the expansion of $\mathfrak{A}$ by this encoding.

Submitted to:
TbiLLC 2025

Part 1: A general approach to the classification of ESO-fragments. In the first part, we analyse the proofs of the existing theorems, in particular the equivalence between FOT(dep) and downward closed ESO. We notice that there is a systematic approach to translate $\varphi(R) \in \text{ESO}$ with property P into a $\varphi^\# \in \text{FOT}(*)$ that consists of four steps:

1. We find a P-specific normal form φ_1 for φ that isolates R in a small subformula β.
2. Using a process from [22], we skolemise φ_1 to get a formula of the form
$$\varphi_2 = \exists f_1 \dots f_n \forall \bar{x}_1 \dots \bar{x}_n (\beta' \wedge \psi')$$
with $\psi' \in \text{FO}$ and a simple $\beta'(R) \in \text{FO}$. Further, f_i only occurs as $f_i\bar{x}_i$ in φ_2.
3. We construct $\varphi_3 \in \text{FOT}(*)$ by replacing all $f_i\bar{x}_i$ by fresh variables z_i and supplement the inner formula by an auxiliary $\gamma \in \text{FOT}(*)$ that is supposed to guarantee that any interpreting team is *function-like*, i.e. factoring in P, it is related to a product of graphs of functions. Overall, φ_3 has the form
$$\varphi_3 = \forall \bar{x}_1 \dots \bar{x}_n \exists z_1 \dots z_n (\gamma \wedge \beta^* \wedge \psi^*),$$
where $\beta^*(R) \in \text{FOT}$ and $\psi^* \in \text{FOT}$.
4. We find a $\beta^\# \in \text{FOT}(*)$ that does not contain R and is equivalent to β^* relative to the property P and function-like teams. We then get
$$\varphi^\# := \forall \bar{x}_1 \dots \bar{x}_n \exists z_1 \dots z_n (\gamma \wedge \beta^\# \wedge \psi^*).$$

While this may serve as a general approach to classification results, several points depend on the specifics of P and $*$, namely the normal form in step 1, the construction of γ in step 3 and the nature of the equivalence in step 4. In particular, γ provides a guideline regarding the atoms in $*$, dependent on P. For example, if P is "downward closure", we can express in FOT(dep) that a team T is a subset of a product of graphs of functions, which is exactly function-likeness up to downward closure. In this spirit, we introduce the *union closed function atom* $\mathbb{F}$ and *upward closed function atom* F, defined by

- $\mathfrak{A}, T \models \mathrm{F}(\bar{x}_1, y_1 | \dots | \bar{x}_n, y_n)$ if and only if there are functions $f_1, \dots, f_n$ in $\mathfrak{A}$ with graphs $g_1, \dots, g_n$ such that $g_1 \times \dots \times g_n \subseteq T(\bar{x}_1, y_1, \dots, \bar{x}_n, y_n)$, and
- $\mathfrak{A}, T \models \mathbb{F}(\bar{x}_1, y_1 | \dots | \bar{x}_n, y_n)$ if and only if there is a family of tuples of functions $(f_1^i, \dots, f_n^i)_i \in I$ with graphs $(g_1^i, \dots, g_n^i)_i \in I$ such that $\bigcup_{i \in I} \left(g_1^i \times \dots \times g_n^i\right) = T(\bar{x}_1, y_1, \dots, \bar{x}_n, y_n)$.

In words, these atoms guarantee function-like behaviour that is adequate for union closure and upward closure, respectively (a more general approach to the definition of new team atoms can be found in [18]). Using the strategy above, we show the following theorems:

Theorem 1. *For sentences,* $\text{ESO} \equiv \text{FOT}(\mathrm{F}) \equiv \text{FOT}(\mathbb{F})$.

Theorem 2. *Every union closed* $\varphi(R) \in \text{ESO}$ *is equivalent to a formula in* $\text{FOT}(\mathbb{F})$*, and vice versa.*

Theorem 3. *Every upward closed* $\varphi(R) \in \text{ESO}$ *is equivalent to a formula in* $\text{FOT}(\mathrm{F})$.

Note that theorem 3 only asserts one direction. The reason is that even first-order literals are not upward closed, and it is therefore impossible to find an straightforward extension of FOT that is equivalent to the upward closed fragment.

Part 2: The convex fragment of ESO. In the second part, we find a team logic corresponding to the convex fragment of ESO. Convexity is of special interest in linguistics, where it arises as a natural feature in the context of lexicalizations (e.g. [21, 10, 19, 20]). In particular, [6] provided an explanation for the non-existence of specific indefinites using a team-semantic convexity argument. There are expressive completeness results for convex extensions of propositional and modal team logics [2]), but the convex fragment of ESO has remained an open question.

More formally, $\varphi(R) \in \mathrm{ESO}$ is quasi-convex[2] if for all non-empty interpretations $R_l^{\mathfrak{A}} \subseteq R^{\mathfrak{A}} \subseteq R_u^{\mathfrak{A}}$, if $\mathfrak{A}\left[R_l^{\mathfrak{A}}/R\right] \models \varphi$ and $\mathfrak{A}\left[R_u^{\mathfrak{A}}/R\right] \models \varphi$, then also $\mathfrak{A}\left[R^{\mathfrak{A}}/R\right]$. As such, it is a refinement of downward and upward closure, and therefore FOT is quasi-convex, as are dependence atoms and upward closed function atoms. $\mathrm{FOT}(\mathrm{dep},\mathrm{F})$, however, is not quasi-convex because neither existential quantification nor disjunction preserves quasi-convexity. We define $\mathrm{FOT}^{\mathrm{cvx}}$ by replacing these operators by convex versions �among ∃ and ∀ convex variants. This is a conservative replacement in the sense that every $\varphi \in \mathrm{FOT}$ is equivalent to its convex version.

$\mathrm{FOT}^{\mathrm{cvx}}(\mathrm{dep},\mathrm{F})$ is quasi-convex and therefore contained in the convex fragment of ESO. For the other direction, we again employ the strategy of part 1. For the normal form in step 1, we notice that for every convex $\varphi \in \mathrm{ESO}$, there is an upward closed φ_u and a downward closed φ_d such that $\varphi \equiv \varphi_u \wedge \varphi_d$, and we find separate normal forms for the two parts of the formula. Then φ_d is translated into a $\varphi_d^\# \in \mathrm{FOT}^{\mathrm{cvx}}(\mathrm{dep})$, and for φ_u, we find a $\varphi_u^\# \in \mathrm{FOT}^{\mathrm{cvx}}(\mathrm{F})$.

Theorem 4. *Every quasi-convex $\varphi(R) \in \mathrm{ESO}$ is equivalent to a formula in $\mathrm{FO}^{\mathrm{cvx}}(\mathrm{dep},\mathrm{F})$, and vice versa.*

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[2] Note that all common team logics have the empty team property, i.e. the empty team satisfies all formulae. This means that "true" convexity (i.e. including empty teams/relations) coincides with downward closure. Therefore, we analyse quasi-convexity as the most natural replacement instead.

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