Downward Closed Guarded Team Logics

Marius Tritschler
TU Darmstadt
Darmstadt, Germany
tritschler@mathematik.tu-darmstadt.de

Team semantics is a natural approach to reason about interdependencies between assignments. Logical formulae are evaluated for *sets of assignments* (called *teams*) instead of single assignments. This enables reasoning about imperfect information, e.g. quantum phenomena [1, 15], or notions of dependence from database theory. Most of the time, first order team logic FOT is augmented with *team atoms* that correspond to dependencies between assignments such as dependence [3], independence [8], inclusion [4] or exclusion [5].

This approach emerged with Väänänen's dependence logic FOT(dep) [19], building on a compositional model-theoretic semantics [14] for independence-friendly logic [13]. Since then, a variety of team logics has been established, including (conditional) independence logic [12] FOT(indep) and inclusion/exclusion logic FOT(inc, exc) [7].

Aside from team atoms, another approach is *hybrid team logic* HTL, which extends FOT by *binders* (↓) that are rooted in extensions of modal logic [17, 9]: let X be a relational variable of arity n, let $\bar{x} = (x_1, \ldots, x_n)$ be a tuple of variables and let ψ be a formula in HTL that may contain X. Then $\downarrow_{\bar{x}} X(\psi)$ is a formula in HTL. For all structures $\mathfrak{A}$ and teams T, we have

$$\mathfrak{A}, T \models \downarrow_{\bar{x}} X(\psi(X)) \quad \text{if} \quad \mathfrak{A}\begin{bmatrix} T(\bar{x}) \\ X \end{bmatrix}, T \models \psi,$$

where $\mathfrak{A}\begin{bmatrix} T(\bar{x}) \\ X \end{bmatrix}$ is the expansion of $\mathfrak{A}$ over X by $T(\bar{x}) = \{t(x_1), \ldots, t(x_n) \mid t \in T\}$. In [18], it is established that HTL and its positive and negative fragments HTL$^+$ and HTL$^-$ are natural fits in the hierarchy of team logics, as can be seen in fig. 1.

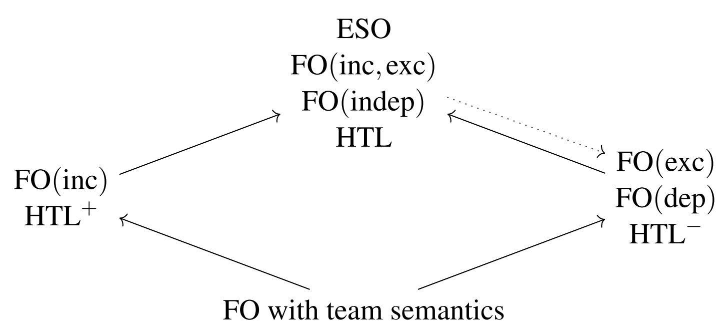

Figure 1: An overview over the hierarchy of team logics. $L \rightarrow L'$ means that L' is at least as expressive as L, with dotted arrows for sentences. Logics that share a node are equiexpressive.

The high expressive power of team logics comes at the cost of a high complexity. In an effort to capture decidable team logics, guarded variants have been considered in [11, 16, 18]. The basic guarded fragment GF of FO [2] is defined by restricting first-order quantification in such a way that formulae

Submitted to:
TbiLLC 2025

can only be evaluated with respect to *guarded tuples*, which are tuples of elements that occur together in some atomic fact. It generalises many of the good model theoretic properties of modal logics; in particular, is has the finite model property [10] and is decidable.

This paper aims to analyze the expressive power of extensions of the team logic variant GTF of GF, building on [18]. A variety of results concerning downward closed logics are provided.

Part 1: Guarded dependence. Guarded dependence logic GTF(dep) does not have the finite model property. We introduce a guarded dependence atom Gdep that, in some sense, expresses functional dependence restricted to guarded patches: for all structures $\mathfrak{A}$ and teams T, $\mathfrak{A}, T \models \text{Gdep}(\bar{x}, \bar{y})$ if for all $t, t' \in T$ with $t(\bar{x}) = t'(\bar{x})$ such that $t(\bar{x})$, $t(\bar{y})$ and $t'(\bar{y})$ are guarded together, we have $t(\bar{y}) = t'(\bar{y})$.

We show that GTF(Gdep) $\equiv$ GTF(exc), mirroring the non-guarded setting. The proof uses a common encoding trick where fresh variables z_1, z_2 are used as a boolean variable that depends on part of the assignment, i.e. they only appear in dependence atoms or in the form $z_1 = z_2$ or $z_1 \neq z_2$. That this is possible even in the guarded setting is non-trivial, and in fact equivalent to the axiom of choice. We further prove that GTF(Gdep) is a proper fragment of GTF(dep).

Part 2: l-local flatness. A family of teams $(T_i)_{i\in I}$ is l-local if for $i \neq j$, the l-neighbourhoods of T_i and T_j in the Gaifman-graph do not overlap. A formula φ is l-locally flat if for all l-local families, we have

$$\bigcup_{i\in I} T_i \models \varphi \quad \Longleftrightarrow \quad \text{f.a. } i \in I : T_i \models \varphi.$$

We notice that every team logic formula that contains the common team atoms, except for constancy atoms, is l-locally flat for some $l \leq \text{qr}(\varphi) + 1$. We show that over naked sets, every such φ can only express statements of the form "there are at most k elements in a structure or team". On the other hand, negative guarded hybrid team logic GHTL$^-$ can express statements of the form "there are at least k distinct elements.".

Theorem 1. *There are sentences in* GHTL$^-$ *that are not expressible in* GTF(inc, dep).

Part 3: GHTL$^-$ **vs.** GESO. Team logics are often classified as fragments of existential second order logic ESO. This means that team logic formulae φ are equivalent to sentences $\varphi^*(R) \in$ ESO in the sense that for all structures $\mathfrak{A}$ and teams T with $\text{dom}(T) = \bar{y}$, we have

$$\mathfrak{A}, T \models \varphi \iff \mathfrak{A}\begin{bmatrix} T(\bar{y}) \\ R \end{bmatrix} \models \varphi^*.$$

In the non-guarded setting, FOT(dep), FOT(exc) and HTL$^-$ are all equally expressive. They stand out because they are equivalent to the fragment of ESO where R appears only negatively, and equivalent to full ESO for sentences. In the guarded setting, we already established that all of these logics have different expressive power. We show that GHTL$^-$ is the best candidate for the non-guarded analogue of these equivalences:

Theorem 2. *Every sentence in* GHTL$^-$ *is equivalent to a sentence of the form* $\exists R_1 \ldots \exists R_n(\psi)$ *with* $\psi \in \text{GF}(R_1, \ldots, R_n)$.

Theorem 3. *A formula in* GHTL$^-$ *is equivalent to a sentence of the form* $\exists R_1 \ldots \exists R_n(\psi)$ *with* $\psi \in \text{GF}(R_1, \ldots, R_n, R)$*, where R appears only negatively in ψ.*

In other words, GHTL$^-$ corresponds to the (downward closed fragment of) prenexed guarded ESO (pre-GESO), which consists of exactly those formulae in GESO that have the above form. The main obstacle in the proof is showing that every sentence $\varphi \in$prenex-GESO is equivalent to a sentence $\varphi^* \in$ GHTL$^-$. For this, we make use of two features of guarded logics:

- In general, guarded quantification can be thought of as moving from guarded patch to guarded patch, while specifying a required overlap between these patches. These moves are always local, except if the overlap is empty, i.e. if the corresponding subformula is a sentence. We can use this to decompose guarded formulae into boolean combinations of subsentence-free formulae.
- Special consideration has to be given to *isolated points*, i.e. singleton connected components in the Gaifman graph of any given structure. We notice that on one hand, the usual encoding tricks are not usable in isolated settings, because all equalities and inequalities are always true and false, respectively. On the other hand, the expressibility of guarded formulae without subsentences is very restricted over isolated points, because the only guarded tuples containing an isolated point a are of the form a^n. Therefore, it is possible to decompose every formula into an isolated and a non-isolated part, and to find characteristic formulae that greatly simplify the isolated parts.

This reduces the problem to non-isolated, subsentence-free φ. Using similar encoding tricks as in part 1, we replace second order quantification with bound teams that appear only negatively.

Part 4: pre-GESO **vs.** GESO**.** As a corollary, we find that GHTL (the guarded variant of HTL) is equivalent to pre-GESO. From previous work, we know that this implies that pre-GESO has the finite model property. We provide a GESO-sentence that is satisfiable and has no finite model, thus showing that contrary to the non-guarded case, pre-GESO is a proper fragment of GESO.

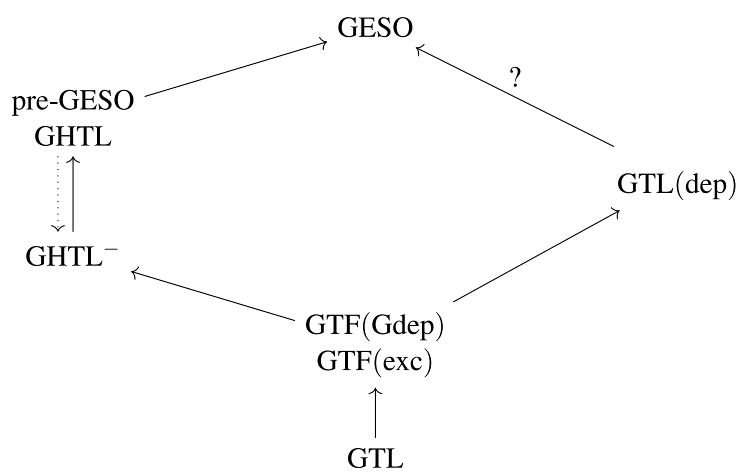

Figure 2: An overview of the classification results in this paper. It remains an open question whether GTF(dep) is a fragment of GESO.

References

[1] R. Albert and E. Grädel. Unifying hidden-variable problems from quantum mechanics by logics of dependence and independence. *CoRR*, abs/2102.10931, 2021.

[2] H. Andréka, I. Németi, and J. van Benthem. Modal languages and bounded fragments of predicate logic. *J. Philos. Log.*, 27(3):217–274, 1998.

[3] W. W. Armstrong. Dependency structures of data base relationships. In J. L. Rosenfeld, editor, *Information Processing, Proceedings of the 6th IFIP Congress 1974, Stockholm, Sweden, August 5-10, 1974*, pages 580–583. North-Holland, 1974.

[4] M. A. Casanova, R. Fagin, and C. H. Papadimitriou. Inclusion dependencies and their interaction with functional dependencies. In J. D. Ullman and A. V. Aho, editors, *Proceedings of the ACM Symposium on Principles of Database Systems, March 29-31, 1982, Los Angeles, California, USA*, pages 171–176. ACM, 1982.

[5] M. A. Casanova and V. M. P. Vidal. Towards a sound view integration methodology. In R. Fagin and P. A. Bernstein, editors, *Proceedings of the Second ACM SIGACT-SIGMOD Symposium on Principles of Database Systems, March 21-23, 1983, Colony Square Hotel, Atlanta, Georgia, USA*, pages 36–47. ACM, 1983.

[6] M. Degano and M. Aloni. Indefinites and free choice. *Natural Language & Linguistic Theory*, 40(2):447–484, May 2022.

[7] P. Galliani. Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information. *Ann. Pure Appl. Log.*, 163(1):68–84, 2012.

[8] D. Geiger, A. Paz, and J. Pearl. Axioms and algorithms for inferences involving probabilistic independence. *Inf. Comput.*, 91(1):128–141, 1991.

[9] V. Goranko. Temporal logic with reference pointers. In D. M. Gabbay and H. J. Ohlbach, editors, *Temporal Logic, First International Conference, ICTL '94, Bonn, Germany, July 11-14, 1994, Proceedings*, volume 827 of *Lecture Notes in Computer Science*, pages 133–148. Springer, 1994.

[10] E. Grädel. On the restraining power of guards. *J. Symb. Log.*, 64(4):1719–1742, 1999.

[11] E. Grädel and M. Otto. Guarded teams: The horizontally guarded case. In M. Fernández and A. Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, volume 152 of *LIPIcs*, pages 22:1–22:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[12] E. Grädel and J. A. Väänänen. Dependence and independence. *Stud Logica*, 101(2):399–410, 2013.

[13] J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In J. E. Fenstad, I. T. Frolov, and R. Hilpinen, editors, *Logic, Methodology and Philosophy of Science VIII*, volume 126 of *Studies in Logic and the Foundations of Mathematics*, pages 571–589. Elsevier, 1989.

[14] W. Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997.

[15] T. Hyttinen, G. Paolini, and J. Väänänen. Quantum team logic and bell's inequalities. *The Review of Symbolic Logic*, 8(4):722–742, June 2015.

[16] M. Lück. *Team logic: axioms, expressiveness, complexity*. PhD thesis, University of Hanover, Hannover, Germany, 2020.

[17] A. Prior. *Past, Present and Future*. Oxford,: Clarendon P., 1967.

[18] M. Tritschler. Guarded Hybrid Team Logics. In A. Murano and A. Silva, editors, *32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)*, volume 288 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 48:1–48:22, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[19] J. A. Väänänen. *Dependence Logic - A New Approach to Independence Friendly Logic*, volume 70 of *London Mathematical Society student texts*. Cambridge University Press, 2007.