An explicit Kuznetsov-Muravitsky enrichment ⋆

Mamuka Jibladze and Evgeny Kuznetsov

TSU Andrea Razmadze Mathematical Institute

1 Introduction

Around 1978, A. V. Kuznetsov introduced the now famous modal extension of the Intuitionistic Propositional Calculus **IPC**, later thoroughly investigated by him and his devoted collaborator A. Yu. Muravitsky in a series of papers. We will use for their system the by now well-established name **KM**.

In the posthumously published paper [3] (only available in Russian) Kuznetsov proved that, for any **IPC**-formulæ A, B,

$$\mathbf{KM} + A \vdash B \iff \mathbf{IPC} + A \vdash B. \tag{K}$$

His proof is essentially proof-theoretic, based on an inductive elimination of the modality from inferences.

On the other hand, from the very beginning Kuznetsov and Muravitsky studied their calculus in parallel with its algebraic semantics, via what they called *Δ-pseudoboolean algebras*. In this paper we will call them **KM**-algebras, following Leo Esakia, who also has an important contribution to their study [1].

A **KM**-algebra is a Heyting algebra $(H, \wedge, \vee, \to, 0, 1)$ together with a unary operation $\varDelta$ satisfying the identities

$$\begin{aligned} x &\leqslant \varDelta x \\ \varDelta x \to x &= x \\ \varDelta x &\leqslant y \vee y \to x. \end{aligned}$$

As established by Kuznetsov and Muravitsky, (K) is equivalent to the statement that every variety of Heyting algebras is generated by reducts of **KM**-algebras. In fact they also showed that (K) implies existence of an embedding of any Heyting algebra H into a **KM**-algebra generating the same variety of Heyting algebras as H. (See [4, p. 53] for these facts.)

As stressed by several people, including Muravitsky himself, it would be highly desirable to have a purely algebraic construction of such an embedding. In [6] he provides one such construction, but his proof that the ambient algebra stays in the same variety is again essentially proof-theoretic, based on a thorough analysis of derivations in various auxiliary modal intuitionistic calculi.

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In this talk, an explicit embedding is addressed, manifestly remaining in the same variety. We do not know whether the extended algebra obtained by us is isomorphic to the one from [6].

Our approach is straightforward: given an element $a \in H$, we adjoin to H a new element $\Delta(a)$ with desired properties freely *in the variety of* H, obtaining an algebra $H\langle\Delta(a)\rangle$ with a homomorphism $H \to H\langle\Delta(a)\rangle$. While it is straightforward to show that such an algebra $H\langle\Delta(a)\rangle$ given by generators and relations exists in our variety and moreover can be characterized by an appropriate universal property, it comes as sort of a black box; in particular, it is not clear whether the corresponding homomorphism $H \to H\langle\Delta(a)\rangle$ is an embedding and, moreover, it is not clear that it preserves all those elements of the form $\Delta(b)$ which happen to already exist in H. And to obtain, iterating this construction in a certain way, an embedding of H into a reduct of a **KM**-algebra, we clearly need to ensure both of these properties. In order to show that this homomorphism is indeed an embedding and does not spoil any other $\Delta(b)$ that might already exist in H, we give an explicit description of the algebra $H\langle\Delta(a)\rangle$. After that we proceed similarly to [6], showing that one can consistently do this for every $a \in H$, obtaining an embedding $H \to H\langle\Delta\rangle$ such that every $a \in H$ possesses a $\Delta(a)$ in $H\langle\Delta\rangle$. Then, again as in [6] we iterate to obtain a countable chain of embeddings $H \to H\langle\Delta\rangle \to H\langle\Delta\rangle\langle\Delta\rangle \to \cdots$ and finally take the union (that is, direct limit) of this chain, which then turns out to be a **KM**-algebra. Thus the only difference of our construction from that in [6] is at the very first step. The point is that, unlike what happens in [6], it is immediate that this first step does not leave the variety of H, and this then obviously remains true at all further steps too.

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Bibliography

[1] Leo Esakia, *The modalized Heyting calculus: a conservative modal extension of the intuitionistic logic*, J. Appl. Non-Classical Logics **16** (2006), no. 3-4, 349–366, DOI 10.3166/jancl.16.349-366.

[2] George Grätzer, *Universal algebra*, Springer, New York, 2008 (second edition), xx+586 pages. DOI 10.1007/978-0-387-77487-9.

[3] A. V. Kuznetsov, *The proof-intuitionistic propositional calculus*, Dokl. Akad. Nauk SSSR **283** (1985), no. 1, 27–30 (Russian). Electronic copy available at https://www.mathnet.ru/rus/dan/v283/i1/p27.

[4] Alexei Y Muravitsky, *The contribution of AV Kuznetsov to the theory of modal systems and structures*, Logic and Logical Philosophy **17** (2008), no. 1-2, 41–58. DOI 10.12775/LLP.2008.004.

[5] Alexei Muravitsky, *Logic KM: a biography*, Leo Esakia on duality in modal and intuitionistic logics, 2014, pp. 155–185. DOI 10.1007/978-94-017-8860-1_7.

[6] Alexei Muravitsky, *On one embedding of Heyting algebras*, 2019. arXiv:1705.02728 [math.LO].