Degree of Kripke Incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$

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1 Introduction

A logic L is Kripke-complete if L is the logic of some class of Kripke-frames. Thomason [11] established the existence of Kripke-incomplete tense logics. Fine [7] and van Benthem [12] gave examples of Kripke-incomplete modal logics. To study Kripke-completeness at a higher level, Fine [7] introduced the degree of Kripke-incompleteness of logics. For any lattice $\mathcal{L}$ of logics and $L \in \mathcal{L}$, the *degree of Kripke-incompleteness* $\mathsf{deg}_{\mathcal{L}}(L)$ *of* L *in* $\mathcal{L}$ is defined as:

$$\mathsf{deg}_{\mathcal{L}}(L) = |\{L' \in \mathcal{L} : \mathsf{Fr}(L') = \mathsf{Fr}(L)\}|.$$ [1]

In general, studying the degree of Kripke-incompleteness in $\mathcal{L}$ amounts to analyzing the equivalence relation $\equiv_{\mathsf{Fr}}$ on $\mathcal{L}$, where $L_1 \equiv_{\mathsf{Fr}} L_2$ iff L_1 shares the same class of frames as L_2, i.e., $\mathsf{Fr}(L_1) = \mathsf{Fr}(L_2)$. The degree of Kripke-incompleteness of L is the cardinality of the equivalence class $[L]_{\equiv_{\mathsf{Fr}}}$ in $\mathcal{L}$.

A celebrated result in this field is the dichotomy theorem for the degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{K})$ by Blok [2]: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. This theorem was proved in [2] algebraically by showing that union splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the consistent normal modal logics of the degree of Kripke-incompleteness 1 and all other consistent logics have the degree $2^{\aleph_0}$. A proof based on relational semantics was given later in [3]. This characterization of the degree of Kripke-incompleteness indicates locations of Kripke-complete logics in the lattice $\mathsf{NExt}(\mathsf{K})$.

Generally, one can always replace the class Fr of all Kripke frames with some proper class $\mathcal{C}$ of mathematical structures, for example, the class MA of all modal algebras or the class Fin of all finite frames. Let $\mathcal{L} = \mathsf{NExt}(\mathsf{K})$. Then we see that $\equiv_{\mathsf{MA}}$ is the identity relation on $\mathcal{L}$ and the equivalence relation $\equiv_{\mathsf{Fin}}$ is a superset of $\equiv_{\mathsf{Fr}}$. Bezhanishvili et al. [1] introduced the notion of the degree of finite model property (FMP) of L in $\mathcal{L}$, which is in fact the cardinality of the equivalence class $[L]_{\equiv_{\mathsf{Fin}}}$. The anti-dichotomy theorem for the degree of FMP for extensions of the intuitionistic propositional logic IPC was proved in [1]: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{Ext}(\mathsf{IPC})$ such that the degree of FMP of L in $\mathsf{Ext}(\mathsf{IPC})$ is κ. It was also shown in [1] that the anti-dichotomy theorem of the degree of FMP holds for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. Degrees of FMP in bi-intuitionistic logics were studied in [6]. Given close connections between bi-intuitionistic logics and tense logics, it is natural to study the degree of Kripke-incompleteness in lattices of tense logics.

Tense logics are bi-modal logics that include a future-looking necessity modality $\Box$ and a past-looking possibility modality $\blacklozenge$, of which the lattices are substantially different from those of modal logics (see [8, 11, 10]). As far as we are aware, there are currently only few results concerning the degree of Kripke-incompleteness in lattices of tense logics. We proved in our recent work [5] that Blok's dichotomy theorem can be generalized to $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$. In this work, we provide a full characterization of the degree of Kripke-incompleteness and the degree of FMP in $\mathsf{NExt}(\mathsf{S4}_t)$. By the characterization, we show that every tense logic $L \in \mathsf{NExt}(\mathsf{S4}_t)$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. It turns out

[1] We denote by $\mathsf{Fr}(L)$ the class of frames validating L.

that in $\mathsf{NExt}(\mathsf{S4}_t)$, iterated splittings, rather than union splittings, are exactly those of the degree of Kripke-incompleteness 1. For more on iterated splittings of lattices of tense and subframe logics, we refer the readers to [13, 9].

2 Main Results

In what follows, we focus on the lattice $\mathsf{NExt}(\mathsf{S4}_t)$ and write $\mathsf{deg}(L)$ and $\mathsf{df}(L)$ for the degree of Kripke-incompleteness and the degree of FMP of L in $\mathsf{NExt}(\mathsf{S4}_t)$, respectively. Let $L_0 \in \mathsf{NExt}(\mathsf{S4}_t)$ and $L_2 \supseteq L_0$. Then L_2 is called a splitting in $\mathsf{NExt}(L_0)$ if there exists $L_1 \in \mathsf{NExt}(L_0)$ such that for all $L' \in \mathsf{NExt}(L_0)$, exactly one of $L' \subseteq L_1$ and $L' \supseteq L_2$ holds. In this case, we write L_0/L_1 for L_2. We call L an iterated splitting if $L = \mathsf{S4}_t/L_1/\cdots/L_n$ for some $L_1, \cdots, L_n \in \mathsf{NExt}(\mathsf{S4}_t)$. Specially, we count $\mathsf{S4}_t$ also as an iterated splitting. Our main result is the following theorem:

Theorem 1. *Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. If L is an iterated splitting, then $\mathsf{df}(L) = 1$. Otherwise $\mathsf{deg}(L) = 2^{\aleph_0}$.*

Note that $\mathsf{deg}(L) \leq \mathsf{df}(L)$, dichotomy theorems for both the degree of FMP and the degree of Kripke-incompleteness for $\mathsf{NExt}(\mathsf{S4}_t)$ follows from Theorem 1. By [8, Theorem 21], $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t\rangle$ and $\langle \mathsf{Log}(\mathfrak{Ch}_1), \mathscr{L}_t\rangle$ are the only two splitting pairs in $\mathsf{NExt}(\mathsf{S4}_t)$.[2] Since the logic $\mathsf{S4}_t/\mathsf{Log}(\mathfrak{Ch}_2)/\mathsf{Log}(\mathfrak{Cl}_3)$ is not a union splitting, Theorem 1 indicates that logics of the degree of Kripke-incompleteness 1 are not necessary union splittings, which shows that Blok's characterization of the degree of Kripke-incompleteness for $\mathsf{NExt}(\mathsf{K})$ can not be generalized to $\mathsf{NExt}(\mathsf{S4}_t)$ directly.

3 Proof Idea

In what follows, we report on the proof idea of Theorem 1 and the main technique used. For definitions of the notations used in the proof, we refer the reader to [4].

Definition 2. *A* Kripke frame *is a pair $\mathfrak{F} = (X, R)$ where $X \neq \varnothing$ and $R \subseteq X \times X$. The* inverse *of R is defined as $\breve{R} = \{\langle v, w\rangle : wRv\}$. For every $w \in X$, let $R[w] = \{u \in X : wRu\}$ and $\breve{R}[w] = \{u \in X : uRw\}$. For every $U \subseteq W$, we define $R[U] = \bigcup_{x\in U} R[x]$ and $\breve{R}[U] = \bigcup_{x\in U} \breve{R}[x]$.*

Let $R^n_\sharp[w]$ be the set of all points which can be reached from w by an $(R\cup\breve{R})$-path of length no more than n. Models, truth and validity of tense formulas are defined as usual.

For each $n \in \omega$ and $\varphi, \psi \in \mathscr{L}_t$, we define the formula $\Delta^n_\psi\varphi$ by:

$$\Delta^0_\psi\varphi = \psi \wedge \varphi \text{ and } \Delta^{k+1}_\psi\varphi = \Delta^k_\psi\varphi \vee \Diamond(\psi \wedge \Delta^k_\psi\varphi) \vee \blacklozenge(\psi \wedge \Delta^k_\psi\varphi).$$

Then the readers can verify that $\mathfrak{M}, w \models \Delta^n\varphi$ if and only if there is an $(R \cup \breve{R})$-path $\langle w_i : i < n\rangle$ such that $w_0 = w$, $\mathfrak{M}, w_{n-1} \models \varphi$ and $\mathfrak{M}, w_i \models \psi$ for all $i < n$. We write $\Delta^k\varphi$ for $\Delta^k_\top\varphi$.

Lemma 3. *Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. Then L is an iterated splitting if and only if $L \in \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$.*

Proof. The key observation is that $\mathsf{Log}(\mathfrak{Cl}_n) = \mathsf{S4}_t/\mathsf{Log}(\mathfrak{Ch}_2)/\mathsf{Log}(\mathfrak{Cl}_{n+1})$ for each $n > 0$. □

Lemma 4. *Let $L \in \mathsf{NExt}(\mathsf{S5}_t)$. Then $\mathsf{df}(L) = 1$.*

Proof. Take any $L' \in \mathsf{NExt}(\mathsf{S4}_t)$ with $\mathsf{Fin}(L') = \mathsf{Fin}(L)$. Then $\mathfrak{Ch}_2 \not\models L'$. Since $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t\rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{S4}_t)$, we have $L' \in \mathsf{NExt}(\mathsf{S5}_t)$. Note that $\mathsf{S5}_t$ has the FMP and is pretabular (see [4]), every extension of $\mathsf{S5}_t$ enjoys the FMP and so $L = L' = \mathsf{Log}(\mathsf{Fin}(L))$. □

[2] For each $n > 0$, We denote the chain of length n and the frame $(n, n \times n)$ by $\mathfrak{Ch}_n$ and $\mathfrak{Cl}_n$, respectively.

To show the second-half of Theorem 1, let $L \in \mathsf{NExt}(\mathsf{S4}_t)$ be an arbitrarily fixed logic such that $L \notin \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$. Take any $\varphi_L \in L \setminus \mathsf{S4}_t$. By the book-construction given in [5], we have

Lemma 5. *There is $\mathfrak{F}_L \in \mathsf{Fin}$ and $w_L, u_L \in X_L$ such that $\mathfrak{F}_L, w_L \not\models \varphi_L$ and $u_L \notin R_\sharp^{\mathsf{md}(\varphi)}[w_L]$.*

For each $I \in \mathcal{P}(\mathbb{Z}^+)$, we define the general frame $\mathbb{F}_I = (X_I, R_I, P_I)$, where the underlying frame $\mathfrak{F}_I = (X_I, R_I)$ is as depicted in Fig.1, and P_I is the tense algebra generated by $\mathcal{P}(X_L)$.

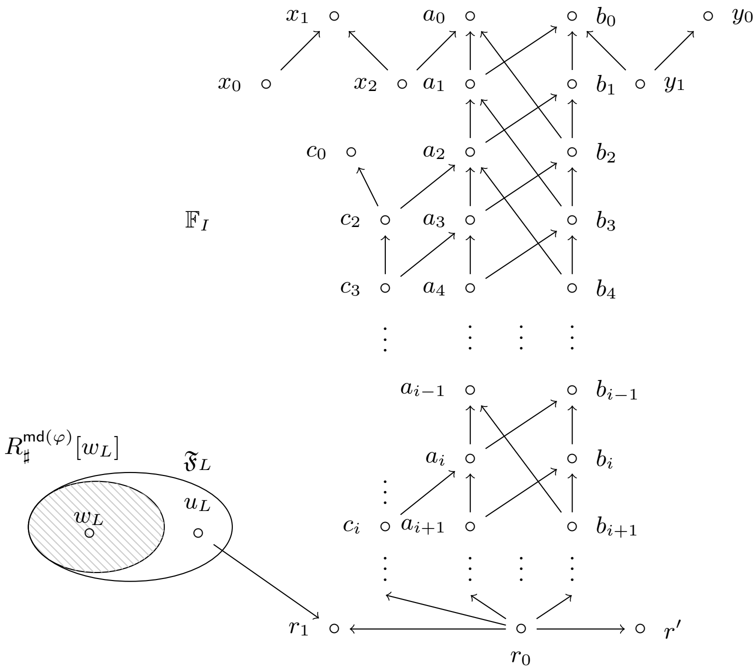

Figure 1: The frame $\mathfrak{F}_I$ where $1 \notin I$ and $2, 3, i \in I$

Let $L_I = L \cap \mathsf{Log}(\mathbb{F}_I)$. To show that $I \neq J$ implies $L_I \neq L_J$, it suffices to prove the following lemma:

Lemma 6. *Let $U = \{a_i : i \in \omega\} \cup \{b_i : i \in \omega\} \cup \{x_0, x_1, x_2, y_0, y_1\}$. For all $u \in U$ and $v \in X_I$,*

(1) $\mathbb{F}_I, u \not\models \varphi_u \to \nabla^k \varphi_L$,

(2) $\mathbb{F}_I \models \neg\varphi_{c_1^j}$ for any $j \notin I$.

The formulas φ_u are defined inductively. Due to limited space, we show only the definition of φ_{x_0} and φ_{a_3}. Let $k > |\mathfrak{F}_L| + 5$. Assume also that $p, p_0, \cdots, p_k \notin \mathsf{Prop}(\varphi_L)$. Then we define

$$\varphi_{x_0} := \Delta^k \neg\varphi_L \wedge \Delta_p^4 \varphi_0 \wedge \nabla^3 \neg\varphi_0 \text{ and } \varphi_{a_3} := \varphi_{AB} \wedge \Diamond\varphi_{a_2} \wedge \Diamond\varphi_{b_2} \wedge \Box\neg\varphi_{b_3},$$

where $\varphi_0 := \neg\mathsf{bd}_k \wedge \blacksquare\neg p$, $\varphi_{AB} = \Box(\varphi_{b_0} \vee \varphi_{b_1} \vee \Diamond\blacklozenge\Diamond\blacklozenge\varphi_{x_0})$ and the formula bd_k is defined in [4, Def.4.3]. By Lemma 6, we see that $\varphi_{c_1^i} \to \nabla^k \varphi_L \in L_J \setminus L_I$, given $i \in I \setminus J$. Thus $I \neq J$ implies $L_I \neq L_J$.

The final step is to show $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ for all $I \in \mathbb{Z}^+$. Key lemmas are as follows:

Lemma 7. *Let $L_1, L_2 \in \mathsf{NExt}(\mathsf{K}_t)$. Then $\mathsf{Fr}_r(L_1 \cap L_2) = \mathsf{Fr}_r(L_1) \cup \mathsf{Fr}(L_2)$.*[3]

Lemma 8. $\mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Iso}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\})$ *for all $I \in \mathbb{Z}^+$.*

[3]We denote by $\mathsf{Fr}_r(L)$ the class of all rooted frames of L.

The key observation for proving Lemma 7 is that if a rooted frame $\mathfrak{F}$ refutes $\varphi_1(\vec{p}) \in L_1$ and $\varphi_2(\vec{q}) \in L_2$, then $\mathfrak{F}$ refutes $\Delta^n\varphi_1 \wedge \varphi_2$ for some $n \in \omega$. For Lemma 8, suppose there exists $\mathfrak{G} \in \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) \setminus \mathsf{Iso}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\})$. Let $k > |\mathfrak{F}_L| + 5$. Then $\mathfrak{G}$ validates $\mathsf{grz}^+, \mathsf{grz}^-, \mathsf{bw}_k^+, \mathsf{bw}_k^+$ and bz_k, which entails that $\mathfrak{G}$ is finite. Let $\mathcal{J}^k(\mathfrak{G})$ be the Jankov-formula of $\mathfrak{G}$. Then $\mathbb{F}_I \not\models \neg\mathcal{J}^k(\mathfrak{G})$ and so $\mathfrak{G}$ is a t-morphic image of $\mathbb{F}_I$. Let $f : \mathbb{F}_I \twoheadrightarrow \mathfrak{G}$. Since $\mathfrak{G} \notin \mathsf{Iso}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\})$, we claim that f does not identify x_0 with other points, i.e., $f^{-1}[f(x_0)] = \{x_0\}$. The proof of this claim will be tedious, so we shall only show here that $x_1 \notin f^{-1}[f(x_0)]$. Suppose $f(x_0) = f(x_1)$. Then for all $y' \in R[f(x_0)]$, there exists $y \in R[x_1]$ such that $f(y) = y'$, which entails $y' = f(x_1) = f(x_0)$. Thus $R[f(x_0)] = \{f(x_0)\}$. Similarly $\breve{R}[f(x_0)] = \{f(x_0)\}$. So $\mathfrak{G} \cong \mathfrak{Ch}_1$, which contradicts the assumption. We can further claim that f does not identify a_0, b_0 and b_1 with any other point. Then by the property of Rieger-Nishimura ladder, we can check that $\mathfrak{G}$ contains an infinite descending chain, which contradicts $\mathfrak{G} \models \mathsf{grz}^-$. Hence $\mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Iso}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\})$.

Finally we are ready to show $\mathsf{Fr}_r(L) = \mathsf{Fr}_r(L_I)$. Since $L \notin \mathsf{NExt}(\mathsf{S5}_t)$ and $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t \rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{S4}_t)$, we have $\mathfrak{Ch}_2 \models L$ and so $\mathsf{Iso}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\}) \subseteq \mathsf{Fr}_r(L)$. By Lemmas 7 and 8, $\mathsf{Fr}_r(L_I) = \mathsf{Fr}_r(L) \cup \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}_r(L)$. Hence $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ for all $I \in \mathbb{Z}^+$.

Note that $I \neq J$ implies $L_I \neq L_J$ for all $I, J \subseteq \mathbb{Z}^+$, we conclude that $\mathsf{deg}(L) = 2^{\aleph_0}$. Note that $L \notin \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$ is also chosen arbitrarily, the proof of Theorem 1 is concluded.

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