Craig Interpolation for Logics of Negative Modality via Cut-Free Sequent Calculus

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This paper investigates the Craig interpolation property for the logics $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ via newly proposed sequent calculi for these three logics. Our calculi are sound and semantically complete to the model-theoretic semantics for $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$, respectively (Theorem 1). Moreover, all the calculi are cut-free and satisfy the subformula property (Theorem 2). We prove the Craig interpolation theorems for $\mathbf{ND}$ and $\mathbf{N}^*$ by making use of the calculus for each logic (Theorem 3). Moreover, we point out that Craig interpolation property for $\mathbf{N}$ does not hold, although Craig interpolation property is usually known as a corollary of the subformula property of a sequent calculus (Theorem 4). We also prove that the syntactic expansions of these three logics by $\bot$ satisfy the Craig interpolation property (Theorem 5).

The logic $\mathbf{N}$, which was proposed by Dŏsen [6], is obtained by adding to the positive intuitionistic propositional logic a negation $\sim$, distinct from intuitionistic negation. In the Kripke semantics for $\mathbf{N}$, a model is defined to be a tuple $\langle W, \leq, C, V\rangle$, where W, $\leq$, and V are the same as the ones in the Kripke semantics for intuitionistic propositional logic, i.e., W is a non-empty set of states, $\leq$ is a partial order on W, and V is a valuation such that $V(p)$ is an upset with respect to $\leq$ for any $p \in \mathsf{Prop}$, and C is a binary relation on W satisfying the following condition:

$$\leq \circ C \subseteq C \circ \leq^{-1},$$

where $R_1 \circ R_2$ stands for the composition of two relations R_1 and R_2, and R^{-1} denotes the inverse relation of a relation R. In this semantics, given a model $M = \langle W, \leq, C, V\rangle$ and a state $w \in W$, the satisfaction relation for $\sim$ is defined as follows:

$$M, w \models \sim\varphi \quad \text{iff} \quad \text{for any } v \in W : wCv \text{ implies } M, v \not\models \varphi.$$

We say that a formula φ is *valid* in a Kripke model M iff for any states w in M, $M, w \models \varphi$. The logic $\mathbf{N}$ is defined as the set of all valid formulas in all Kripke models. One feature of $\mathbf{N}$ is that the negation $\sim$ is weaker than intuitionistic negation and even than minimal negation. Došen [6] also observed that a systematic framework, where $\mathbf{N}$ is placed as the weakest logic, can be proposed like modal logics (cf. [1]). Due to this similarity to modal logics, the negation "$\sim$" can be categorized into a *negative modality*. Došen investigated $\mathbf{N}$ and its extensions also in [7], and a systematic approach to logics with negative modalities were taken in [24, 19]. The logic $\mathbf{ND}$ is defined as the set of all valid formulas in all Kripke models satisfying *seriality*, described as follows: for any state w in the Kripke model, there is a state v in the model such that wCv. Thus, $\mathbf{ND}$ is an extension of $\mathbf{N}$.

The logic $\mathbf{N}^*$, proposed by Cabalar, Odintsov, and Pearce [2], is an extension of $\mathbf{ND}$, implying that it is also an extension of $\mathbf{N}$. Although the Kripke semantics can be provided for $\mathbf{N}^*$, simpler semantics called the *star semantics* was proposed in [2]. In the star semantics, a model called a *star model* is defined. A star model is defined to be a tuple $\langle W, \leq, *, V\rangle$, where W, $\leq$, and V are the same as the ones in the Kripke semantics for intuitionistic propositional logic, and $*$ is a function from W to W satisfying the following condition for any $w, v \in W$:

$$w \leq v \text{ implies } v^* \leq w^*.$$

In this semantics, given a model $M = \langle W, \leq, *, V\rangle$ and a state $w \in W$, the satisfaction relation for $\sim$ is defined as follows:

$$M, w \models \sim\varphi \quad \text{iff} \quad M, w^* \not\models \varphi.$$

We say that a formula φ is *valid* in a star model M iff for any states w in M, $M, w \models \varphi$. The logic $\mathbf{N}^*$ is defined as the set of all valid formulas in all star models. One feature of $\mathbf{N}^*$ is that it validates all the de Morgan laws but does not validate the double negation law. As pointed out in [18], the propositional logic $\mathbf{HYPE}$, studied in [17, 13], can be defined as the set of all valid formulas in all star models satisfying the *involution condition*, described as follows: for any state w in the star model, $w = w^{**}$. Thus, $\mathbf{N}^*$ is a logic strictly weaker than $\mathbf{HYPE}$. The logic $\mathbf{HYPE}$ validates not only all the de Morgan laws but also the double negation law.

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When it comes to the proof-theoretic perspective, Hilbert systems for $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ were already provided. The Hilbert system for $\mathbf{N}$, provided by Došen [6], is obtained by adding the following axiom and rule to the Hilbert system for positive intuitionistic propositional logic:

$$(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi),$$
$$\text{From } \varphi \to \psi, \text{ we may infer } \sim\psi \to \sim\varphi.$$

The Hilbert system for $\mathbf{ND}$, studied in [6, 7], is obtained by adding the following axiom to that for $\mathbf{N}$:

$$\sim(\varphi \to \varphi) \to \psi.$$

The Hilbert system for $\mathbf{N}^*$, studied in [2, 8], was obtained by adding the following axioms to that for $\mathbf{ND}$:

$$\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi),$$
$$\sim((\varphi \to \varphi) \to \psi).$$

On the other hand, to the authors' best knowledge, there is no proof theory other than Hilbert system for each of these logics. Especially, there is no sequent calculus for any of them. Although a sequent calculus for a logic with negative modalities was investigated in [5, 12], the logic has four negative modalities in its syntax and each rule for a negative modality depends on the existence of other negative modalities. Moreover, intuitionistic implication $\to$ does not exist in the syntax of [5, 12].

This paper newly proposes sequent calculi for $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$. The sequent calculus for $\mathbf{N}$, denoted by $\mathsf{G}(\mathbf{N})$, is obtained by adding to the positive propositional fragment of Maehara's [14] multi-succedent sequent calculus mLJ for intuitionistic logic the following rule for $\sim$:

$$\frac{\varphi \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\varphi}\ (\sim N),$$

where $\sim\Delta = \{\sim\chi \mid \chi \in \Delta\}$. Note that in this rule φ must exist, i.e., the antecedent of the upper sequent cannot be $\emptyset$, although Δ can be $\emptyset$. The sequent calculus for $\mathbf{ND}$, denoted by $\mathsf{G}(\mathbf{ND})$, is obtained by replacing $(\sim N)$ with the following rule:

$$\frac{\Sigma \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Sigma}\ (\sim ND),$$

where Σ is either singleton or $\emptyset$, $\sim\Sigma = \{\sim\chi \mid \chi \in \Sigma\}$, and $\sim\Delta = \{\sim\chi \mid \chi \in \Delta\}$. In this rule both Γ and Δ can be $\emptyset$, respectively. On the other hand, Σ cannot contain more than one formula. The sequent calculus for $\mathbf{N}^*$, denoted by $\mathsf{G}(\mathbf{N}^*)$, is obtained by replacing $(\sim ND)$ with the following rule:

$$\frac{\Gamma \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Gamma}\ (\sim N^*),$$

where $\sim\Gamma = \{\sim\chi \mid \chi \in \Gamma\}$ and $\sim\Delta = \{\sim\chi \mid \chi \in \Delta\}$. Note that in this rule Γ can contain more than one formula.

Theorem 1. For any formula φ, all the following hold:

(1) $\Rightarrow \varphi$ is derivable in $\mathbf{G}(\mathbf{N})$ iff φ is valid in all Kripke models,

(2) $\Rightarrow \varphi$ is derivable in $\mathbf{G}(\mathbf{ND})$ iff φ is valid in all serial Kripke models,

(3) $\Rightarrow \varphi$ is derivable in $\mathbf{G}(\mathbf{N}^*)$ iff φ is valid in all star models.

All of (1), (2), and (3) can be shown by proving that $\mathsf{G}(\mathbf{N})$, $\mathsf{G}(\mathbf{ND})$, and $\mathsf{G}(\mathbf{N}^*)$ are equipollent to the Hilbert system for $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ described above, respectively.

In addition, we can prove the cut-elimination theorem for all the calculi.

Theorem 2. Let $\Lambda \in \{\mathbf{N}, \mathbf{ND}, \mathbf{N}^*\}$. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\Lambda)$, then there is a derivation in $\mathsf{G}(\Lambda)$ whose root is $\Gamma \Rightarrow \Delta$ with no application of (Cut).

For each of the calculi, this theorem is shown purely syntactically by employing the notion of "extended cut rule", which were used in [10, 20]. Except for the argument for the rule of $\sim$, our argument is the same as the one for the cut elimination for mLJ. Note that as in the argument for the cut elimination for mLJ, we need to appeal to the inversion of several rules in order to deal with the right rule for implication, as noted in [23]. The argument for the rule of $\sim$ is straightforward. As the corollary of Theorem 2, we can obtain the subformula property of $\mathsf{G}(\mathbf{N})$, $\mathsf{G}(\mathbf{ND})$, and $\mathsf{G}(\mathbf{N}^*)$.

As far as the authors know, the Craig interpolation properties for $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ are open problems. Since the Craig interpolation theorem is usually proved as a corollary of the subformula property of a calculus for the intended logic, it is natural to ask whether the theorem holds or not in $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ in this stage. Note that since neither $\top$ nor $\bot$ exists in the syntax for the logics, we need to slightly modify the formulation of Craig interpolation theorem. Actually, the modified version of the theorem is essentially the same as the original formulation by Craig [3, 4] himself. Thus, an interesting question is whether this modified version of the Craig interpolation theorem holds or not in $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$.

For $\mathbf{ND}$ and $\mathbf{N}^*$, we can answer affirmatively.

Theorem 3. Let $\Lambda \in \{\mathbf{ND}, \mathbf{N}^*\}$. If $\varphi \to \psi$ is derivable in $\mathsf{G}(\Lambda)$, one of the following holds:

- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) \neq \emptyset$, then there is a formula χ such that both $\varphi \to \chi$ and $\chi \to \psi$ are derivable in $\mathsf{G}(\Lambda)$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$,
- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) = \emptyset$, then either $\varphi \Rightarrow$ or $\Rightarrow \psi$ is derivable in $\mathsf{G}(\Lambda)$.

Our argument for Theorem 3 is based on a modified version of Maehara's [15] method, provided by Seki [22] in order to prove the Craig interpolation property for classical and intuitionistic logic and some substructural logics without $\top$ or $\bot$. On the other hand, since both $\mathsf{G}(\mathbf{ND})$ and $\mathsf{G}(\mathbf{N}^*)$ are multi-succedent sequent calculi and the argument in [22] for intuitionistic logic is based on the single-succedent sequent calculus, we have to revise Seki's modified version of Maehara's method. For $\mathsf{G}(\mathbf{ND})$, we appeal to the argument in [11], originally proposed for bi-intuitionistic propositional logic, where the notion of *normal partitions* are employed. For $\mathsf{G}(\mathbf{N}^*)$, we use the method called "Mints' symmetric interpolation method", proposed in [16]. This method was also employed in [21] to prove the Craig interpolation property for a bi-intuitionistic tense logic. The basic idea of this method is to generalize the notion of an interpolant in order to deal with a multi-succedent sequent calculus.

On the other hand, we give the negative answer to the Craig interpolation for $\mathsf{G}(\mathbf{N})$.

Theorem 4. All of the following hold:

- $\Rightarrow {\sim}(q \to q) \to {\sim} p$ is derivable in $\mathsf{G}(\mathbf{N})$,
- $\mathsf{Prop}({\sim}(q \to q)) \cap \mathsf{Prop}({\sim} p) = \emptyset$,
- neither ${\sim}(q \to q) \Rightarrow$ nor $\Rightarrow {\sim} p$ is derivable in $\mathsf{G}(\mathbf{N})$.

This theorem implies the failure of the Craig interpolation property for $\mathbf{N}$. The underivability of ${\sim}(q \to q) \Rightarrow$ in $\mathsf{G}(\mathbf{N})$ seems somewhat awkward, but by the cut elimination for the system, we can observe that it is impossible to derive a sequent of the form $\Gamma \Rightarrow$ in general.

We can also deal with the syntactic expansions of $\mathbf{N}$, $\mathbf{ND}$, and $\mathbf{N}^*$ by the falsum constant $\bot$ whose satisfaction clause is given by $M, w \not\models \bot$. Let us use $\mathbf{N}_\bot$, $\mathbf{ND}_\bot$, and $\mathbf{N}^*_\bot$ to denote these three logics, respectively. The sequent calculi $\mathsf{G}(\mathbf{N}_\bot)$, $\mathsf{G}(\mathbf{ND}_\bot)$, and $\mathsf{G}(\mathbf{N}^*_\bot)$ are obtained by adding the following axiom to $\mathsf{G}(\mathbf{N})$, $\mathsf{G}(\mathbf{ND})$, and $\mathsf{G}(\mathbf{N}^*)$, respectively:

$$\frac{}{\bot \Rightarrow}\ (\bot \Rightarrow).$$

All of $\mathbf{N}_\bot$, $\mathbf{ND}_\bot$, and $\mathbf{N}^*_\bot$ satisfy the Craig interpolation property.

Theorem 5. Let $\Lambda \in \{\mathbf{N}_\bot, \mathbf{ND}_\bot, \mathbf{N}^*_\bot\}$. If $\varphi \to \psi$ is derivable in $\mathsf{G}(\Lambda)$, then there is a formula χ such that both $\varphi \to \chi$ and $\chi \to \psi$ are derivable in $\mathsf{G}(\Lambda)$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

For $\mathsf{G}(\mathbf{N}_\bot)$ and $\mathsf{G}(\mathbf{ND}_\bot)$, the argument in [11], where the notion of normal partitions is used, is employed. For $\mathsf{G}(\mathbf{N}^*_\bot)$, Mints' symmetric interpolation method is used.

One further direction of the research is to check the Craig interpolation property for other logics with negative modalities. Especially, the Craig interpolation property for $\mathbf{HYPE}$ is considered to be a next step. The sequent calculus for $\mathbf{HYPE}$ was provided in [9] but the restriction on the rule (Cut) to the analytic cut rule was not discussed. It was revealed in [21] that the restriction to the analytic cut rule is sufficient to apply Mints' symmetric interpolation method. Thus, what should be done first is to check whether this restriction is possible in the sequent calculus for $\mathbf{HYPE}$, provided in [9].

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2001.

[2] Pedro Cabalar, Sergei Odintsov, and David Pearce. Logical foundations of well-founded semantics. In Patrick Doherty, John Mylopoulos, and Christopher Welty, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the Tenth International Conference (KR2006)*, Book 1. American Association for Artificial Intelligence, 2006.

[3] William Craig. Linear reasoning: a new form of the Herbrand-Gentzen theorem. *The Journal of Symbolic Logic*, 22(3):250–268, 1957.

[4] William Craig. Three uses of the Herbrand-Gentzen theorm in relating model theory and proof theory. *The Journal of Symbolic Logic*, 22(3):269–285, 1957.

[5] Adriano Dodó and João Marcos. Negative modalities, consistency and determinedness. *Electronic Notes in Theoretical Computer Science*, 300:21–45, 2014.

[6] Kosta Došen. Negation as a modal operator. *Reports on Mathematical Logic*, 20:15–27, 1986.

[7] Kosta Došen. Negation in the light of modal logic. In Dov M Gabbay and Heinrich Wansing, editors, *What is Negation?*, volume 13 of *Applied Logic Series* (*APLS*), pages 77–86. Springer Dordrecht, 1999.

[8] Sergey Drobyshevich and Sergei Odintsov. Finite model property for negative modalities (in Russian). *Siberian Electronic Mathematical Reports*, 10:1–21, 2013.

[9] Martin Fischer. Sequent calculi for the propositional logic of HYPE. *Studia Logica*, 110:643–677, 2022.

[10] Ryo Kashima. *Mathematical Logic* (in Japanese). Asakura Publishing Company, 2009.

[11] Tomasz Kowalski and Hiroakira Ono. Analytic cut and interpolation for bi-intuitionistic logic. *The Review of Symbolic Logic*, 10(2):259–283, 2017.

[12] João Lahav, Orim Marcos and Yoni Zohar. Sequent systems for negative modalities. *Logica Universalis*, 11:342–382, 2017.

[13] Hannes Leitgeb. HYPE: a system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 48:305–405, 2018.

[14] Shôji Maehara. Eine Darstellung der intuitionistischen Logik in der klassischen. *Nagoya Mathematical Journal*, 7:45–64, 1954.

[15] Shôji Maehara. Craig's interpolation theorem (in Japanese). *Sûgaku*, 12(4):235–237, 1961.

[16] Giorgi Mints. Interpolation theorems for intuitionistic predicate logic. *Annals of Pure and Applied Logic*, 113(1-3):225–242, 2001.

[17] Grigore C Moisil. Logique modale. *Disquisitiones mathematicae et Physicae*, 2:3–98, 1942.

[18] Sergei Odintsov and Heinrich Wansing. Routley star and hyperintensionality. *Journal of Philosophical Logic*, 50:33–56, 2021.

[19] Takuro Onishi. Substructural negations. *The Australasian Journal of Logic*, 12(4), 2015.

[20] Hiroakira Ono and Yuichi Komori. Logics without the contraction rule. *The Journal of Symbolic Logic*, 50(1):169–201, 1985.

[21] Hiroakira Ono and Katsuhiko Sano. Analytic cut and Mints' symmetric interpolation method for bi-intuitionistic tense logic. In David Fernández-Duque, Alessandra Palmigiano, and Sophie Pinchinat, editors, *Advances in Modal Logic*, volume 14, pages 601–623. 2022.

[22] Takahiro Seki. On the proof of the interpolation theorem without using constants. *Japan Advanced Institute of Science and Technology Research Report: IS-RR-2001-013*, pages 1–23, 2001.

[23] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2012.

[24] Chunlai Zhou. Perp and star in the light of modal logic. `https://festschriften.illc.uva.nl/D65/zhou.pdf`. 2004.