The Rule Dichotomy Property via Stable Canonical Rules

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1 Introduction

A multi-conclusion inference rule Γ/Δ is called *admissible* in a logic L if for any substitution σ, if $\sigma\gamma \in L$ for all $\gamma \in \Gamma$ then $\sigma\delta \in L$ for some $\delta \in \Delta$. Friedman [7] asked whether the admissibility of a given inference rule in IPC is decidable. Rybakov showed that this is the case for IPC and a large class of transitive modal and superintuitionistic logics (see [10] for a comprehensive overview and references). However, the decidability of admissibility in the basic modal logic K is a long-standing open question.

An *admissible base* is a set of admissible rules from which every admissible rule is derivable. Since the admissibility in a decidable logic is Π^0_1 by definition, the existence of a Σ^0_1 (i.e., recursively enumerable) admissible base implies the decidability of admissibility. Jeřábek [8] introduced a new method to construct admissible bases and establish the decidability of admissibility. Given a logic L, the method consists of two parts: finding a class of rules that can axiomatize all rules over L and providing an admissible base for those selected rules over L. The latter is done concurrently by proving the *rule dichotomy property over* L for the class of rules, and thus yields an additional result that L has the *rule dichotomy property.*

Definition 1. Let L be a modal logic and $\mathcal{R}$ be a class of rules. $\mathcal{R}$ has the *rule dichotomy property over* L if every rule in $\mathcal{R}$ is either L-admissible or L-equivalent to an assumption-free rule. L has the *rule dichotomy property* if every rule is L-equivalent to a set of rules which are either L-admissible or assumption-free.

Note that if there is a class of rules that axiomatizes all rules over L and has the rule dichotomy property over L, then L has the rule dichotomy property. This property is often obtained in the processing of showing the decidability of admissibility, indicating a strong sign of the decidability of admissibility. Moreover, it has consequences on the admissibility in the extensions ([8, Cor 4.5]).

Canonical rules, a generalization of Zakharyaschev's canonical formulas [11], were used in [8] for several transitive modal logics and IPC. However, these rules do not axiomatize all rules over K. Stable canonical rules, originally introduced in [1], do axiomatize all rules over K, and are applied to IPC, K4, and S4 in [4]. Both provide an admissible base and prove the rule dichotomy property of the logics in question.

In this paper, we explore the possibility of generalizing this method with stable canonical rules to non-transitive modal logics wK4 and K. We show that stable canonical rules have the rule dichotomy property over wK4 while not over K. As a partial result toward the decidability of admissibility on K, we provide decidable sufficient conditions for stable canonical rules to be K-admissible or K-inadmissible and discuss some examples.

We assume familiarity with modal logic, modal spaces, and multi-conclusion modal rules; see, e.g., [5], [6], and [1, Sec 2] for details. While all results are stated using the language of relational structures, one can always formulate them in algebraic terms.

2 The rule dichotomy property in wK4

Definition 2. Let $X = (X, R)$ and $Y = (Y, Q)$ be modal spaces, $\mathcal{D} \subseteq Y$, and $f : X \to Y$ be a continuous map. We call f *stable* if xRy implies $f(x)Qf(y)$. We say that f satisfies the *closed domain condition (CDC)* for $\mathcal{D}$ if for any $D \in \mathcal{D}$, $Q[f(x)] \cap D \neq \emptyset$ implies $f[R[x]] \cap D \neq \emptyset$. We write $f : X \twoheadrightarrow_{\mathcal{D}} Y$ if f is an onto stable map satisfying CDC for $\mathcal{D}$ and $X \twoheadrightarrow_{\mathcal{D}} Y$ if there is such an f.

Stable canonical rules are introduced in [1]. For a finite modal space (i.e., a finite Kripke frame) F and $\mathcal{D} \subseteq F$, the validity of the stable canonical rule $\rho(F, \mathcal{D})$ has the following characterization (the dual of [1, Thm 5.4]): for any modal space X, $X \not\models \rho(F, \mathcal{D})$ iff $X \twoheadrightarrow_{\mathcal{D}} F$.

We will write $\Box^{\leq 1}\varphi$ for $\varphi \wedge \Box\varphi$. For $l, m, n \in \omega$, let $S_n^{l,m}$ and T_n^m be the following rules, where we follow the convention $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

$$S_n^{l,m} \qquad \frac{[\bigwedge_{i=1}^{l}(\Box v_i \to v_i) \wedge \bigwedge_{i=1}^{m} \Box(r_i \to \Box(r_i \vee \Box^{\leq 1} q))] \to \bigvee_{i=1}^{n} \Box p_i}{\Box^{\leq 1} q \to p_1, \ \cdots \ , \ \Box^{\leq 1} q \to p_n}$$

$$T_n^{m} \qquad \frac{\bigwedge_{i=1}^{m}(\Diamond r_i \to \Diamond(r_i \wedge \Box^{\leq 1} q)) \to \bigvee_{i=1}^{n} \Box p_i}{\Box^{\leq 1} q \to p_1, \ \cdots \ , \ \Box^{\leq 1} q \to p_n}$$

Generalizing the proof for K4 in [4], we obtain the rule dichotomy property for stable canonical rules over wK4.

Theorem 3. The following are equivalent:

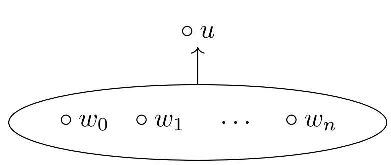
1. $\rho(F, \mathcal{D})$ is wK4-admissible,
2. $\rho(F, \mathcal{D})$ is derivable from $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$,
3. $\rho(F, \mathcal{D})$ is not wK4-equivalent to an assumption-free rule.

However, it remains open whether all rules can be axiomatized by stable canonical rules over wK4. In particular, we lack a continuous filtration for wK4, continuous in the sense that when applied to modal spaces, the canonical projection map is continuous. Although the finite model property of wK4 was proved in [2] and recently in [9], their filtrations are non-standard, and it is unclear whether they are continuous. An alternative approach would be to use canonical rules for wK4; an algebraic proof of the finite model property of wK4 can be found in [3].

3 The rule dichotomy property in K

Contrary to wK4 and transitive modal logics, the rule dichotomy property over K fails for stable canonical rules. In fact, there are infinitely many stable rules (introduced in [1] as stable canonical rules with $\mathcal{D} = \emptyset$) that are neither K-admissible nor K-equivalent to an assumption-free rule.

For $n \in \omega$, let F_n be the following modal space. Points in the circle form a cluster; they all see u and are not seen by u.



The next theorem follows from Thm 7.

Theorem 4. For any $n \in \omega$, $\rho(F_n, \emptyset)$ is K-inadmissible.

Moreover, by constructing a counterexample showing that the validity of $\rho(F_n, \emptyset)$ is not preserved by closed upset and applying [8, Prop 2.5], we obtain the following theorem.

Theorem 5. For any $n \in \omega$, $\rho(F_n, \emptyset)$ is not K-equivalent to any assumption-free rule.

Thus, stable canonical rules fail to have the rule dichotomy property over K. This supports the possibility that K does not have the rule dichotomy property, which aligns with the remark "the rule dichotomy is a very strong property which is unlikely to hold for a substantial class of logics" in [8]. Moreover, this implies that it is impossible to prove a similar result as Thm 3 for K, so we cannot apply Jeřábek's method of establishing the decidability of admissibility with stable canonical rules to K.

4 Some partial results on the admissibility in K

In this final section, we present some sufficient conditions for stable canonical rules to be admissible or K-inadmissible. Note that these conditions are all decidable.

For a modal space $X = (X, R)$, a point $x \in X$ is called a *sharp root* of X if xRy for all $y \in X$. A subset $\mathcal{D} \subseteq \mathsf{P}(F)$ is called *trivial* if $\mathcal{D} = \emptyset$ or $\mathcal{D} = \{\emptyset\}$. Combinaorial proofs on modal spaces show the following.

Theorem 6. Let $F = (F, Q)$ be a finite modal space and $\mathcal{D} \subseteq \mathsf{P}(F)$. If one of the following conditions is *not* satisfied, then $\rho(F, D)$ is K-admissible.

1. F has a sharp root r such that $\forall D \in \mathcal{D}(D \neq \emptyset \to r \in D)$ or $\exists w \in F(w \neq r \wedge wQw \wedge wQr)$.
2. For any $\mathcal{D}' \subseteq \mathcal{D}$ and $d \in \bigcup \mathcal{D}'$, there is a path in $\bigcup \mathcal{D}'$ from d to a maximal point in $\bigcup \mathcal{D}'$.

The following theorem is shown by finding a $\varphi \notin \mathsf{K}$ such that $/\varphi$ is derivable from $\rho(F, D)$.

Theorem 7. Let $F = (F, Q)$ be a finite modal space with a sharp root r and $\mathcal{D} \subseteq \mathsf{P}(F)$ be trivial. Then, $\rho(F, D)$ is K-inadmissible.

As an application, we provide some examples.

Example 8. Let $\mathcal{S}_{\mathsf{K}}$ be the rule system generated by $\{/\varphi : \varphi \in \mathsf{K}\}$. For a rule system $\mathcal{S}$, let $\Lambda(\mathcal{S})$ be the logic of $\mathcal{S}$, namely, $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$. The axiomatizations below are from [1, Sec 8]. We abbreviate $\rho(F, \emptyset)$ as $\rho(F)$.

1. A stable rule $\rho(\mathcal{F}, \emptyset)$ is K-admissible iff $\mathcal{F}$ has no sharp root.
2. Let **Rooted** be the class of finite rooted modal spaces. Then $\mathcal{S}(\textbf{Rooted}) = \mathcal{S}_{\mathsf{K}} + \rho(\,) + \rho(\circ\,\circ) + \rho(\circ\!\rightarrow\!\propto\!\leftarrow\!\circ)$. The three rules are all K-admissible since none of the corresponding spaces has a sharp root. This confirms that K is complete with respect to **Rooted**.
3. $\mathsf{KD} = \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\bullet) + \rho(\circ\!\rightarrow\!\bullet))$. Since $\mathsf{K} \subsetneq \mathsf{KD}$, at least one of the two rules must be K-inadmissible. Thm 6 and 7 tell us that $\rho(\bullet)$ is K-admissbile and $\rho(\circ\!\rightarrow\!\bullet)$ is not. Similarly, regarding $\mathsf{KT} = \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\bullet) + \rho(\propto\!\leftrightarrow\!\bullet))$, $\rho(\bullet)$ is K-admissbile and $\rho(\propto\!\leftrightarrow\!\bullet)$ is not.
4. Up to equivalence, there are exactly 1 K-inadmissible stable canonical rule $\rho(F, \mathcal{D})$ with $|F| = 1$ and 5 K-inadmissible stable canonical rule $\rho(F, \mathcal{D})$ with $|F| = 2$.

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