Axiomatization and Decidability of Tense Information Logic

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Modal information logic (MIL), introduced by van Benthem [7], models information flow using possible worlds semantics of modal logic by introducing additional modalities. Recently, Knudstorp [4] axiomatized MIL with a supremum modality on poset frames, capturing the 'fusion' or 'merge' of information states. Since infima naturally represent the common refinement of such states in posets, a natural question arises: Can the axiomatization of MIL be extended to a version that includes both modalities—supremum and infimum—which we call tense information logic (TIL)? And what axioms govern the relation between the two modalities [8]?

In this abstract there are two main results we will cover: First, following the outline and proofs of [4] we give an axiomatization of TIL on posets, surprisingly showing that the only axioms needed for linking $\langle sup \rangle$ and $\langle inf \rangle$ are the standard tense-logic axioms. Second, we prove that TIL enjoys the FMP w.r.t. a generalized class of frames, thereby establishing its decidability.

In related work, modal logics with supremum and infimum operators have been studied on richer algebraic structures, such as lattices [11,12,3] and semilattices [6]. In particular, Wang & Wang [11,12] obtain completeness over lattices for a hybrid language with supremum and infimum operators as well as nominals.

1 Preliminaries

We begin by giving the definitions that are essential for our discussion of TIL:

Definition 1. *Given a countable set of propositional letters* $\mathbf{P}$*, we define the language* $\mathcal{L}_T$ *of tense information logic using two binary modalities* $\langle sup \rangle$ *and* $\langle inf \rangle$ *by the following BNF grammar:*

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle sup \rangle\varphi\psi \mid \langle inf \rangle\varphi\psi$$

Definition 2. *A (Kripke) poset model for* $\mathcal{L}_T$ *is a triple* $M = (W, \leq, V)$*, where* W *is a set,* $\leq$ *is a partial order and* V *is a valuation* $V : \mathbf{P} \to \mathcal{P}(W)$*.*

Given a poset frame ($\mathfrak{F} = (W, \leq)$), we say that x is the supremum of $\{y, z\}$ if x is an upper bound of y and z and the least such. We say that x is the infimum of $\{y, z\}$ if x is a lower bound of y and z and the greatest such.

The interpretation of a formula φ at a state $x \in W$ is then defined recursively as follows:

$$\begin{array}{rll} \mathfrak{M}, x \nVdash \bot & & \\ \mathfrak{M}, x \Vdash p & \text{iff} & x \in V(p) \\ \mathfrak{M}, x \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, x \nVdash \varphi \\ \mathfrak{M}, x \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi \\ \mathfrak{M}, x \Vdash \langle sup \rangle\varphi\psi & \text{iff} & \text{there exist } y, z \in W \text{ such that } \mathfrak{M}, y \Vdash \varphi,\ \mathfrak{M}, z \Vdash \psi \text{ and } x = sup\{y, z\} \\ \mathfrak{M}, x \Vdash \langle inf \rangle\varphi\psi & \text{iff} & \text{there exist } y, z \in W \text{ such that } \mathfrak{M}, y \Vdash \varphi,\ \mathfrak{M}, z \Vdash \psi \text{ and } x = inf\{y, z\} \end{array}$$

Using these semantics, we can define the past- and forward-looking unary modalities of temporal logic as $P\varphi := \langle sup \rangle\varphi\top$, $F\varphi := \langle inf \rangle\varphi\top$, the dual H as $\neg P \neg$ and G as $\neg F \neg$ [1,4]. Validity in a frame is defined in the usual way.

Definition 3. *Let TIL be the frame-based logic of all* $\mathcal{L}_T$ *validities on poset frames, i.e.*

$$TIL := \{\varphi \in \mathcal{L}_T : \text{ for every poset model } M = (W, \leq, V) \text{ and every } x \in W : M, x \Vdash \varphi\}$$

⋆ This work forms part of the first author's master's thesis, currently in progress, under the supervision of the second author and Nick Bezhanishvili. We thank Nick Bezhanishvili and the anonymous referees for helpful comments.

We extend the axiomatization of MIL given in [4] to define the following syntactically defined logic:

Definition 4. *Let* ***TIL*** *be the least normal modal logic in the language* $\mathcal{L}_T$ *containing the following axioms:*

$$\begin{array}{rl} \textit{(Re.)} & (p \wedge q \to \langle sup\rangle pq) \wedge (p \wedge q \to \langle inf\rangle pq) \\ \textit{(4)} & (PPp \to Pp) \wedge (FFp \to Fp) \\ \textit{(Co.)} & ((\langle sup\rangle pq \to \langle sup\rangle qp) \wedge ((\langle inf\rangle pq \to \langle inf\rangle qp) \\ \textit{(Dk1)} & (p \wedge \langle sup\rangle qr) \to \langle sup\rangle pq \\ \textit{(Dk2)} & (p \wedge \langle inf\rangle qr) \to \langle inf\rangle pq \\ \textit{(Sy.)} & (p \to GPp) \wedge (p \to HFp) \end{array}$$

The only truly new axiom, that expresses the relation between $\langle sup\rangle$ and $\langle inf\rangle$, is the standard temporal axiom (Sy.) [10]. That this axiom is to be included is expected, since the temporal operators P, F, G and H are all definable.

Soundness of **TIL** with respect to *TIL* (i.e., **TIL** $\subseteq$ *TIL*) follows by a standard argument.

2 Results

We now state our primary result, which confirms that our axiomatization correctly captures the logic *TIL*:

Theorem 5 (Completeness). ***TIL*** *is strongly complete w.r.t TIL.*

In the canonical model, the preorder $\leq_{pre}$ induced by the canonical relations C_{sup} and C_{inf} does not form a poset in which these relations are the true supremum and infimum (see Remark A.0.1 of [4]). Consequently, the canonical model lacks the properties required to establish the necessary truth lemma for the completeness proof. To overcome this, we make use of the step-by-step method (see [1]).

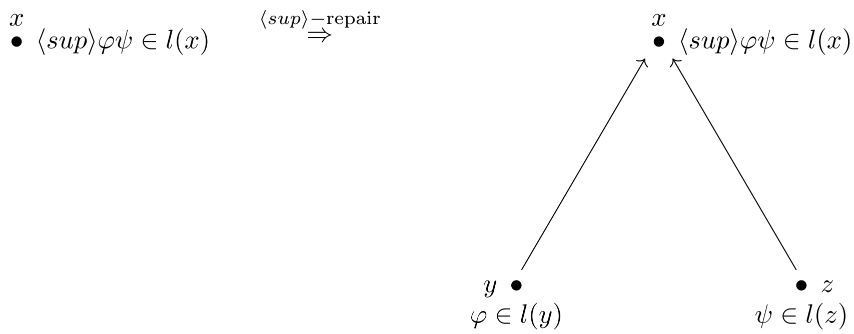
To prove that syntactic consistency implies satisfiability (and hence completeness), we recursively construct a model by repairing so-called 'defects'. We define a labeling function l that assigns to each point in our structure a maximally consistent set (MCS) of the canonical model. Since our goal is to prove the truth lemma for the specific set of formulas we start with, we may and will assign the same MCS to different points, to ensure that other points satisfy the formulas dictated by their MCS-label.

As illustrated in the figures below, a defect can occur in one of four forms:

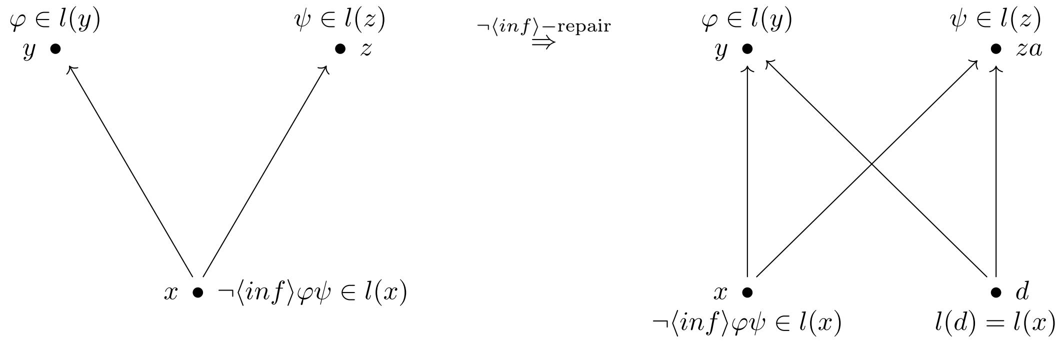
- x is labeled by the MCS Γ and $\langle sup\rangle\varphi\psi \in \Gamma = l(x)$, but $x \nVdash \langle sup\rangle\varphi\psi$, or $\langle inf\rangle\varphi\psi \in \Gamma = l(x)$, but $x \nVdash \langle inf\rangle\varphi\psi$.
- x is labeled by the MCS Γ and $\neg\langle sup\rangle\varphi\psi \in \Gamma = l(x)$, but $x \Vdash \langle sup\rangle\varphi\psi$, or $\neg\langle inf\rangle\varphi\psi \in \Gamma = l(x)$, but $x \Vdash \langle inf\rangle\varphi\psi$.

We resolve these issues through four repair lemmas. Below, we graphically illustrate two of these lemmas to provide intuition.

$\langle sup\rangle$-repair lemma



¬⟨*inf*⟩-repair lemma



Having established completeness of TIL on poset frames, we immediately obtain its completeness with respect to a broader class of frames, namely preorders. $\mathfrak{M} = (W, \leq, V)$ is a preorder model if $\leq$ is a preorder on W. In this case, suprema and infima need not be unique, but can come in clusters. To account for this, we denote that x is a supremum or infimum of $\{y, z\}$ by writing $x \in \sup\{y, z\}$ or $x \in \inf\{y, z\}$, respectively. The semantics of the modal operators are changed accordingly.

Definition 6.

$$TIL_{pre} = \{\varphi \in \mathcal{L}_T : \textit{ for every preorder model } \mathfrak{M} = (W, \leq, V) \textit{ and every } x \in W : \mathfrak{M}, x \Vdash \varphi\}$$

Theorem 7. *Tense information logic interpreted over preorders is the same as its interpretation over posets. That is,* $TIL_{pre} = TIL =$ ***TIL****.*

Proof. Since every poset frame is also a preorder frame, $TIL_{pre} \subseteq TIL$. Furthermore, the soundness proof of **TIL** for poset frames carries over to preorder frames without significant changes, implying **TIL** $\subseteq TIL_{pre}$.

This result provides yet another example of how modal information logics cannot differentiate preorders from posets (see also [5]).

Building on our completeness result, we prove decidability of *TIL*, thus resolving an open problem posed in [9].

Theorem 8. *TIL is decidable*

Since the logic does not have the finite model property (FMP) w.r.t. preorders, nor posets (see [4] Proposition 1.7), we show that it has the FMP w.r.t. a generalized class of frames. In this setting, a general frame is given by a tuple (W, C_{sup}, C_{inf}), where W is a set and C_{sup} and C_{inf} are arbitrary ternary relations on W.

Let $\tilde{\mathcal{C}}$ denote the class of general frames that satisfy the first order correspondents of the axioms of **TIL**. Because poset frames are special cases of general frames and standard frame-correspondence arguments give the equivalence

$$(W, C_{sup}, C_{inf}) \Vdash \mathbf{TIL} \iff (W, C_{sup}, C_{inf}) \in \tilde{\mathcal{C}},$$

it follows that **TIL** is sound and strongly complete with respect to $\tilde{\mathcal{C}}$, so that **TIL** $= Log(\tilde{\mathcal{C}})$.

We then show by a filtration argument that any formula $\chi \notin$ **TIL** (i.e. invalid on some, possibly infinite, $\tilde{\mathcal{C}}$-frame) can be shown to be invalid on a finite $\tilde{\mathcal{C}}$-frame. This implies that **TIL** has the FMP w.r.t. this class of frames, i.e. **TIL** $= Log(\tilde{\mathcal{C}}_F)$.

Since **TIL** is a finitely axiomatisable normal modal logic that has the FMP, we conclude that *TIL* is decidable. Note that from $TIL = TIL_{pre}$, we immediately get the same result for TIL_{pre}.

References

1. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
2. Burgess, J.P.: Basic tense logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic: Volume II: Extensions of Classical Logic, pp. 89–133. Springer, Dordrecht (1984). https://doi.org/10.1007/978-94-009-6259-02
3. Bezhanishvili, N., Dmitrieva, A., de Groot, J., Moraschini, T.: Positive modal logic beyond distributivity. Annals of Pure and Applied Logic **175**, 103374 (2024). https://doi.org/10.1016/j.apal.2023.103374
4. Knudstorp, S.: Modal Information Logics: Axiomatizations and Decidability. Journal of Philosophical Logic **52**(6), 1723–1766 (2023). https://doi.org/10.1007/s10992-023-09724-5
5. Knudstorp, S.: The Modal Logic of Minimal Upper Bounds. Language, Logic, and Computation. 14th International Tbilisi Symposium, TbiLLC 2023, Telavi, Georgia, forthcoming
6. Knudstorp, S.B.: Modal Information Logics. Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam (2022).
7. van Benthem, J.: Modal Logic as a Theory of Information. In: Copeland J. (Ed.) Logic and Reality. Essays on the Legacy of Arthur Prior, pp 135–168. Oxford University Press, Oxford (1997). https://doi.org/10.1093/oso/9780198240600.003.0008
8. van Benthem, J.: Truth Maker Semantics and Modal Information Logic. Draft manuscript (2017). Available at https://eprints.illc.uva.nl/id/eprint/1590/
9. van Benthem, J.: Constructive agents. Indagationes Mathematicae **29**(1), 23–35 (2018). https://doi.org/10.1016/j.indag.2017.10.004
10. Rescher, N., Urquhart, A.: Temporal Logic. Springer, Vienna (1971)
11. Wang, X., Wang, Y.: Modal Logics over Lattices. In: Annals of Pure and Applied Logic, **176**(4). Elsevier BV (2025). https://doi.org/10.1016/j.apal.2025.103553
12. Wang, X., Wang, Y.: Tense Logics over Lattices. In: Logic, Language, Information, and Computation, pp 70–87. Springer International Publishing (2022). https://doi.org/10.1007/978-3-031-15298-6_5