A COALGEBRAIC SEMANTICS FOR FISCHER-SERVI LOGIC

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The semantics of modal logics lend themselves quite naturally to a coalgebraic representation. Given a frame (X, R), one can conceive of the relation R as a map $R[-]$ sending a point to its set of successors. Thus, a frame can in general be conceived of as an object and a map; this is the basis for the *coalgebraic approach* to logic (see for example chapter 9 of [7]). As the primary example, the category of classical Kripke frames is isomorphic to the category of coalgebras on the powerset functor, where (X, R) is represented as the coalgebra $(X, R[-] : X \to \mathcal{P}(X))$.

While classical and positive modal logics have well-known coalgebraic semantics, their intuitionistic counterpart has remained somewhat elusive. Recently, [2] introduced a technique to extend coalgebras for positive modal logic into coalgebras for intuitionistic modal logic (IML). This was shown for the $\Box$-only fragment of IML, though it was remarked there that similar techniques apply for many other settings, such as extending the work of [3] on the coalgebraic semantics for Dunn-style positive modal logic to the intuitionistic case.

A central limitation of such an approach lies in the fact that one needs to start with a positive modal logic, and find its least intuitionistic extension; this leaves out several interesting cases, such as the logic **IK** due to [4]. In this paper we show how the approach of [2] can be modified to yield coalgebraic completeness for this and other related logics. We will develop the coalgebraic representation for Esakia spaces, and derive consequences for intuitionistic Kripke frames over image-finite posets, using the same methods as in [2].

Definition 1. A *positive general frame* (PGF) is a quadruple $(X, \leq, R, A)$ where $(X, \leq)$ is a Priestley space, $R \subseteq X \times X$ is a relation such that for any $x \in X$, $R[x]$ is closed, and A is the set of clopen upsets of $(X, \leq)$. We say that a positive general frame is an *intuitionistic general frame* (IGF) if $(X, \leq)$ is Esakia.

It will be usual to demand a lot more of the relation R, depending on the modalities. The key to turning a coalgebra for a PGF into one for an IGF lies in the dual action of generating the free Heyting algebra off of a distributive lattice. This is done via the functor $\mathcal{V}_G$, which is the right adjoint to the inclusion of Esakia spaces into Priestley spaces (for details on this construction and its analogue for image-finite Kripke frames, see [1]). This allows us to take the coalgebras for any positive modal logic $\mathcal{L}$, and extend them into coalgebras for its smallest intuitionistic extension.

Thus, applying this to the coalgebraic semantics of, for example, Dunn's positive modal logic (see, e.g., [3]) would give us the representation for an intuitionistic modal logic with both modal operators and several compatibility axioms. However, one of the most prominent intuitionistic modal logics does not fit into this framework:

Definition 2 (Axiomatisation of **IK**)**.** An algebra $(H, \wedge, \vee, \to, \Box, \Diamond, \top, \bot)$ is called an **IK**-algebra if $(H, \wedge, \vee, \to, \top, \bot)$ is a Heyting algebra, and it satisfies the following modal axioms:

1. $\Box\top = \top$	2. $\Diamond\bot = \bot$
3. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
A. $\Diamond(a \to b) \leq \Box a \to \Diamond b$	**B.** $\Diamond a \to \Box b \leq \Box(a \to b)$

TABLE 1. Axioms for **IK**-algebra. [5]

The logic associated with **IK**-algebras is often called **IK** or *Fischer-Servi logic*; its semantics differ from other intuitionistic modal logics, and it has some strong claims to providing a good intuitionistic analogue for classical modal logic – for example, there exists a standard translation for it into intuitionistic first order logic (see [6] for details). Given the interaction between modalities and implications in axioms **A** and **B**, there is no ostensible positive reduct for which **IK** is the smallest intuitionistic extension.

Recall that given a set (X, R), we write $\Diamond_R U = \{x \in X : \exists y, xRy \text{ and } y \in U\}$ and $\Box_R U = X - \Diamond_R(X - U)$. We will also make use of bimodal general frames:

Definition 3. A $\Box\Diamond$-*frame* is a triple $(X, R_\Box, R_\Diamond)$ such that X is an Esakia space, and the following conditions hold:

- $R_\Box[x]$ is a closed upset
- $R_\Diamond[x]$ is a closed downset
- If U is a clopen upset, then $\Diamond U$ and $\Box U$ are clopen upsets

Definition 4. A ***FS***-*frame* is an intuitionistic general frame (X, R) such that the following conditions hold:

(T1) $R[x]$ is closed
(T2) $R[\uparrow x]$ is a closed upset
(T3) If U is a clopen upset, then $\Diamond_R U$ and $\Box_{(\leq \circ R)} U$ are clopen upsets
(T4) $R[x] = R[\uparrow x] \cap \downarrow R[x]$

By the results in [5], there is a duality between **FS**-frames and **IK**-algebras. Thus, an **FS**-frame is a $\Box\Diamond$-frame where $R := R_\Box \cap R_\Diamond$ and the following conditions hold:

(i) $R_\Diamond = \downarrow(R_\Box \cap R_\Diamond)$ and
(ii) $R_\Box = \leq \circ (R_\Box \cap R_\Diamond)$

Our goal is to find an endofunctor on Esakia spaces such that its coalgebras correspond in a natural way to **FS**-frames. This can be done in steps, starting by representing $\Box\Diamond$-frames; these correspond to the smallest intuitionistic extension of positive modal logic satisfying **IK** axioms (1-4). Thus, our first step is to turn a $\Box\Diamond$-frame into a coalgebra.

Definition 5. Let $\mathcal{V}$ be a Priestley endofunctor taking a space X to a space $\mathcal{V}(X) \subseteq \mathcal{P}(X)$. The *Vietoris topology* on $\mathcal{V}(X)$ is given by sets of the form $[U] = \{C \in \mathcal{V}(X) | C \subseteq U\}$ and $\langle V \rangle = \{C \subseteq \mathcal{V}(X) | C \cap V \neq \emptyset\}$, for U, V clopen.

Definition 6. The functors $\mathcal{V}^\uparrow(X)$ and $\mathcal{V}^\downarrow(X)$ (called the *upper Vietoris space* and *lower Vietoris space* of X) are defined as follows:

(1) $\mathcal{V}^\uparrow(X) = \{C \subseteq X | C \text{ is a closed upset}\}$, ordered by reverse inclusion, with the topology given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen upsets of X;
(2) $\mathcal{V}^\downarrow(X) = \{C \subseteq X | C \text{ is a closed downset }\}$, ordered by inclusion, with the topology given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen downsets of X.

A $\Box\Diamond$-frame $(X, R_\Box, R_\Diamond)$ has two relations, where $R_\Box[x]$ is a closed upset and $R_\Diamond[x]$ a closed downset, from which we obtain the following:

Theorem 7. *There is a 1-1 correspondence between* $\Box\Diamond$*-frames and coalgebras for the Priestley endofunctor* $(\mathcal{V}^\uparrow \times \mathcal{V}^\downarrow)$. *By the results in [2], the categories of* $\Box\Diamond$*-frames and* $\mathbf{CoAlg}(V_G(\mathcal{V}^\uparrow \times \mathcal{V}^\downarrow))$ *are isomorphic.*

As an intuition for the algebraically-minded, if D_X is the dual distributive lattice to X, then $\mathcal{V}^\uparrow$ dually corresponds to generating the free distributive lattice over $\{\Box a | a \in D_X\}$ and quotienting over the normality axioms for $\Box$. Similarly, $\mathcal{V}^\downarrow$ does so over $\{\Diamond a | a \in D_X\}$. Thus, **IK** axioms (1-4) are taken care of. Note that in the space $\mathcal{V}^\uparrow(X)$, elements $\Box a$ correspond to the clopen upsets $[U]$, and in the space $\mathcal{V}^\downarrow(X)$, elements $\Diamond a$ correspond to the clopen upsets $\langle U \rangle$.

To achieve a coalgebraic representation for **FS**-frames, we will need to look at subspaces which satisfy conditions (i) and (ii). We start by identifying the following subspace of $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$:

Definition 8. Let $FS_1(X) = \{(D, C) \in \mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X) : C = \downarrow (D \cap C)\}$.

Proposition 9. *$FS_1(X)$ is the Priestley subspace of $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$ consisting of the sets of pairs (D, C) such that $(D, C) \in \mathcal{V}^{\uparrow}(X) \times \langle U \to V \rangle \cap [U] \times \mathcal{V}^{\downarrow}(X) \implies (D, C) \in \mathcal{V}^{\uparrow}(X) \times \langle V \rangle$.*

That is, for any $(D, C) = (R_{\Box}[x], R_{\Diamond}[x])$, then $R_{\Diamond}[x] = \downarrow(R_{\Box}[x] \cap R_{\Diamond}[x])$ for any x if and only if axiom **A** is dually satisfied, corresponding to condition (i). Notice that if X is an Esakia space, then we have implications between clopen upsets, so for axiom **A** we only need to add the modal operators via the functor $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$, and dually quotient over **A** by taking the subspace $FS_1(X)$. However, axiom **B** involves implications between modal formulas, so before we can quotient, we must add elements of the form $\{a \to b | a, b \in D_{FS_1(X)}\}$. We do this via the functor V_r forming the set of closed and rooted subsets of X, and take $V_r(FS_1(X))$ (for details, see [1]). Now we may look at a subspace satisfying (ii):

Definition 10. Let $FS_2(X) = \{C \in V_r(FS_1(X)) | \forall (D, E) \in C$, $y \in D$ and $y \leq z$, there exists $(D', E') \geq (D, E)$ in C such that $z \in D' \cap E'\}$

Proposition 11. *FS_2 is the Priestley subspace of $V_r(FS_1)$ for which axiom **B** dually holds, i.e. $\forall U, V \in ClopUp(X)$, $C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup [V] \times \mathcal{V}^{\downarrow}(X)] \implies C \in [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$.*

We now have a 1-1 correspondence between **FS**-frames (X, R) and maps $\alpha : X \to FS_2(X)$ which commute with the root map $r : FS_2(X) \to FS_1(X)$. To turn these into coalgebras for an appropriate endofuntor on Esakia spaces, we look at the composition $\mathcal{V}_G^r \circ FS_2(X)$. Here, the r superscript specifies that we preserve the first layer of implications, which we added previously; maps preserving this layer are called *r-open*. This leads us to our main result:

Theorem 12. *The following are in 1-1 correspondence:*

(1) ***FS**-frames (X, R),*
(2) *r-open Priestley maps $\alpha : X \to FS_2(X)$, and*
(3) *Coalgebras for the Esakia endofunctor $\mathcal{V}_G^r(FS_2(X))$*

Our method extends to image-finite Kripke frames as well using the P_g and P_G functors from [1], in an analogous way to the case in [2], by using Birkhoff duality in place of Priestley duality. We then use this representation to derive several results that follow from coalgebraic completeness. Specifically, we characterize bisimulations and the construction of free **IK**-algebras. In further work, we also intend to look at frame conditions of special interest, namely those of monadic intuitionistic propositional calculus (MIPC), and intuitionistic S4.

Bibliography

[1] Rodrigo Nicolau Almeida. "Colimits and Free Constructions of Heyting Algebras through Esakia Duality". In: *arXiv preprint* (2024).

[2] Rodrigo Nicolau Almeida and Nick Bezhanishvili. "A Coalgebraic Semantics for Intuitionistic Modal Logic." In: *arXiv preprint* (). arXiv: `2406.10649 [math.LO]`.

[3] Guram Bezhanishvili, John Harding, and Patrick J. Morandi. "Remarks on hyperspaces for Priestley spaces". In: *Theoretical Computer Science* 943 (2023), pp. 187–202. DOI: `10.1016/j.tcs.2023.07.015`.

[4] Giselle Fischer Servi. "Axiomatizations for some intuitionistic modal logics". In: *Rendiconti del Seminario Matematico della Università di Torino* 42 (1984), pp. 179–194.

[5] Alessandra Palmigiano. "Dualities for some intuitionistic modal logics". In: *Liber Amicorum for Dick de Jongh.* Institute for Logic, Language and Computation, 2004, pp. 151–167.

[6] Alex K. Simpson. “The Proof Theory and Semantics of Intuitionistic Modal Logic”. PhD thesis. University of Edinburgh, 1994.

[7] Yde Venema. “Algebras and Coalgebras”. In: *Studies in Logic and Practical Reasoning*. Vol. 3. Elsevier, 2007, pp. 331–426.