HMS-Style Duality for Residuated Lattices

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In the semantics of modal logic, topological dualities between categories of modal algebras and categories of modal spaces have played a central role in the development of the field [4, 5]. On one hand, these dualities have pragmatic purpose; they unify the available algebraic and model theoretic methods used to address questions about modal logics and their semantics. On the other hand, dualities have an explanatory purpose since they offer alternative perspectives on, and clarify why, the methods of one semantics or another are successful.

A paradigm case of the sort of insight duality can provide is the clarity it casts on canonical model completeness proofs with respect to Kripke frames. It is known that the canonical model of a normal modal logic comes with a natural modal space topology and that this modal space is the dual of Lindenbaum algebra of the same logic. This fact can then be coupled with what is sometimes called *d-persistence*, which is the property that if a formula is valid in a modal space, then it is also valid in the underlying Kripke frame of that modal space (see [4, 5]). Canonical model style Kripke completeness proofs can then be understood in virtue of three distinct steps: algebraic completeness, duality, and d-persistence. For more details we refer the reader to [4] and [5] for a proof of the Sahlqvist Completeness Theorem using topological duality - one of the most celebrated results in modal logic.

For non-classical logics, and in particular logics with algebraic semantics given in terms of residuated lattices, topological dualities have received varying degrees of attention. These dualities are often obtained by extending a duality for the lattice reducts of the algebras in question. For example, dualities for distributive residuated lattices tend to build on Priestley duality. A very successful instance of this strategy is the duality between Heyting algebras and Esakia Spaces. Another more general method has been to modify Priestley spaces with a ternary relation [13]. For not-necessarily-distributive residuated lattices, dualities are obtained by building on dualities for bounded lattices. An illustrative example is Allwein and Dunn's [1] extension of Urquhart's topological representation of lattices to a representation of various residuated algebras [12]. Another are the dualities given in terms of canonical extensions [8, 7]. These are early examples, but a great number of dualities for various classes of residuated lattices could be listed, many building on various dualities for bounded lattices [2].

In this paper we present a novel topological duality for not-necessarily-distributive residuated lattices by modifying a recent duality for bounded lattices established by Bezhanishvili et al. in [3]. In much the same way that the various dualities mentioned above link algebraic semantics to model theoretic semantics, our duality establishes a natural connection between the algebraic semantics of substructural logics and a sort of frame semantics for substructural logics originating in the work of Ono and Komori [11], Humberstone [10], and Došen [6]. This connection will allow us to the further generalize the completeness theorems in [11, 10, 6] and to gain insight into the success of the canonical model style proofs present in these papers. In particular we adapt the notion of Π_1-persistence from [3] and show that the canonical model style proofs in [11, 10, 6] can be explained by an analysis similar to the one given for canonical modal logics above.

The contribution of the work contained here is therefore twofold. On the one hand we provide a technical contribution: a novel duality for residuated lattices. On the other hand, our project is expository: we wish to explain the semantics of [11, 10, 6] by providing a fine grained description of the relevant notion of canonicity in terms of algebraic completeness, topological duality, and Π_1-persistence.

1 Duality for Residuated Lattices

Recall that a *(bounded) residuated lattice ordered groupoid* or *$r\ell$-groupoid* $\mathbf{G} = (G, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ is an algebra where (i) $(G, \wedge, \vee, \top, \bot,)$ is a bounded lattice, (ii) $(G, \cdot, e)$ is a unital groupoid with e as identity, and (iii) $a \cdot b \leq c$

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A unital groupoid is a algebra $(G, \cdot, e)$ where $e \cdot a = a = a \cdot e$ for all $a \in G$. No other conditions are placed on $\cdot$ and in particular $\cdot$ is not necessarily associative.

iff $b \leq a\backslash c$ iff $a \leq c/b$. A *residated lattice* is an $r\ell$-groupoid where $\cdot$ is associative. An *$r\ell$-groupoid homomorphism* $f : \mathbf{G} \to \mathbf{H}$ is a function which preserves each of the algebraic operations. The category **RLG** is the category of $r\ell$-groupoids with $r\ell$-groupoid homomorphisms.

The starting point for our topological dualities for residuated lattices and more generally $r\ell$-groupoids is a recent topological duality for bounded lattices established Bezhanishvili et al. in [3]. It is shown in [3] that Hoffmann, Mislove, and Stralka's topological duality for semilattices can be restricted to the case of lattices in a natural way [9]. Bezhanishvili et al. define an *L-space* $\mathbf{X} = (X, \curlywedge, 1, \tau)$ to be a compact, 0-dimensional topological semilattice that satisfies the HMS-separation axiom (see below) and has the property that for all clopen filters U and V, the filter $U \triangledown V := \{z \in X \mid \exists x, y(x \in U \ \&\ y \in V \ \&\ x \curlywedge y \leq z)\}$ is also clopen in $\mathbf{X}$. We define as usual that $x \leq y$ iff $x \curlywedge y = x$.

(HMS-separation) If $x \not\leq y$, then there is a clopen filter U such that $x \in U$ and $y \notin U$.

The morphisms $f : \mathbf{X} \to \mathbf{Y}$ between L-spaces, called *L-space morphisms*, are continuous semilattice homomorphisms that additionally satisfy the following two constraints: (i) for all $x \in X$, $fx = 1_Y$ iff $x = 1_X$ and (ii) for all $x', y' \in Y$ and $z \in X$, If $x' \curlywedge_Y y' \leq fz$, then there are $x, y \in X$ such that $x \curlywedge_X y \leq z$ and $x' \leq fx$ and $y' \leq fy$. In [3] it is shown that the category of bounded lattices and lattice homomorphisms is dually equivalent to the category of L-spaces with L-space morphisms.

Our topological duality for $r\ell$-groupoids, residuated lattices, and various subclasses of these algebras are obtained with respect to classes of structured topological spaces that we call NRL-spaces, which modify L-spaces by adding an additional binary operation $\otimes$ and an identity element for that operation, ε.

Definition 1.1. An NRL-*space* $\mathbf{X} = (X, \curlywedge, \otimes, 1, \varepsilon, \tau)$ is a semilattice ordered groupoid $(X, \curlywedge, 1, \otimes, \varepsilon)$ such that $(X, \curlywedge, 1, \tau)$ is an L-space and:

(1) If U, V are clopen filters, then $U \circ_X V = {\uparrow}\{x \otimes y \mid x \in U \ \&\ y \in V\}$, $U\backslash_X V = \{y \mid \forall x \in U(x \otimes y \in V)\}$, and $V/_X U = \{x \mid \forall y \in U(x \otimes y \in V)\}$ are too.

(2) ${\uparrow}\varepsilon$ is clopen,

(3) $x \otimes y \leq z$ iff for all clopen filters U, V, if $x \in U$ and $y \in V$, then $z \in U \circ_X V$,

(4) (a) $x \otimes \varepsilon = x = \varepsilon \otimes x$

(b) $x \otimes 1 = 1 = 1 \otimes x$, and

(c) $x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z)$ and $(y \curlywedge z) \otimes x = (y \otimes x) \curlywedge (z \otimes x)$.

Henceforth we will write $\mathcal{F}_{clp}(X)$ to denote the set of all clopen filters of $\mathbf{X}$. Generally, we also write $\mathcal{F}(X)$ to denote the set of all filters of $\mathbf{X}$.

Lemma 1.1. (Between Spaces and Algebras)

(1) For any NRL-space $\mathbf{X}$, the algebra $\mathbf{G_X} = (\mathcal{F}_{clp}(X), \cap, \triangledown, X, \{1\}, \circ_X, \backslash_X, /_X, {\uparrow}\varepsilon)$ is an $r\ell$-groupoid.

(2) For any rl-groupoid $\mathbf{G}$, we have that $\mathbf{X_G} = (\mathcal{F}(G), \cap, G, \otimes_G, \tau)$ is an NRL-space where τ is generated by the subbase $\mathcal{S}_G = \{\phi_{\mathbf{G}}(a) \mid a \in G\} \cup \{(\phi_{\mathbf{G}}(a))^c \mid a \in G\}$ with $\phi_{\mathbf{G}}(a) = \{x \in \mathcal{F}(G) \mid a \in x\}$ and $x \otimes_G y = {\uparrow}\{a \cdot b | \ a \in x \ \&\ b \in y\}$.

In accordance with the previous lemma, we will henceforth denote the $r\ell$-groupoid of clopen filters of a given NRL-space $\mathbf{X}$ by $\mathbf{G_X}$. Conversely, the NRL-space defined on the filters of a given $r\ell$-groupoid $\mathbf{G}$ will be denoted by $\mathbf{X_G}$.

Morphism between NRL-spaces are defined as L-space morphisms that satisfy a collection of additional conditions.

Definition 1.2. An NRL-*space morphism* $f : \mathbf{X} \to \mathbf{Y}$ is an L-space morphism that satisfies the following conditions:

($\otimes$-forth) $f(x) \otimes' f(y) \leq f(x \otimes y)$,

($\otimes$-back) If $x' \otimes' y' \leq f(z)$, then there are $x, y \in X$ such that $x' \leq fx$, $y' \leq fy$, and $x \otimes y \leq z$,

($/$-back) if $fx \otimes' y' \leq z'$, then there are $y, z \in X$ such that $y' \leq fy$, $fz \leq z'$, and $x \otimes y \leq z$,

($\backslash$-back) if $x' \otimes' fy \leq z'$, then there are $x, z \in X$ such that $x' \leq fx$, $fz \leq z'$, and $x \otimes y \leq z$,

(ε-forth) $\varepsilon' \leq f(\varepsilon)$, and

(ε-back) if $\varepsilon' \leq fx$, then $\varepsilon \leq x$.

The category of NRL-spaces together with NRL-space morphisms is denoted by **NRLSp**. Together with the following we arrive at our duality theorem.

Lemma 1.2. (Between Morphisms)

(1) $f : \mathbf{G} \to \mathbf{H}$ is $r\ell$-groupoid homomorphism, then $f^{-1} : \mathbf{X_H} \to \mathbf{X_G}$ is a continuous NRL-morphism.

(2) If $f : \mathbf{X} \to \mathbf{Y}$ is an NRL-space morphism, then $f^{-1} : \mathbf{G_Y} \to \mathbf{G_X}$ is an $r\ell$-groupoid homomorphism.

Recall that a topological spaces (X, τ) is 0-dimensional if τ has a basis of clopens.

Theorem 1.1. (Duality) The category **NRLSp** is dually isomorphic to the category **RLG**.

Proof. Both of the morphisms $\phi_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}_{\mathbf{X}_{\mathbf{G}}}$ and $\eta_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}_{\mathbf{G}_{\mathbf{X}}}$ are isomorphisms in their respective categories. $\phi_{\mathbf{G}}$ was defined in Lemma 1.1 and $\eta_{\mathbf{X}}(x) = \{U \in \mathcal{F}_{clp}(X) \mid x \in U\}$ □

Theorem 1.1 can be extended in many ways. In particular, we obtain duality for residuated lattices.

Corollary 1.1. The category of residuated lattices **RLat** is dually isomorphic the the category **RLSp** of NRL-spaces in which $\otimes$ is associative.

2 Applications and Explications

As discussed in the introduction, the duality presented above has a tight connection to the semantics of sub-structural logics developed in [11, 6, 10] and more recently in [14]. The key insight that made completeness theorems with respect to these semantics successful was the treatment of additive disjunction $\vee$ as an intensional connective. In their most general form, and within some degree of variation, the frames used in [11, 6, 10] are semilattice ordered groupoids which satisfy conditions (4a-c) in Definition 1.1. Given a frame $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon)$, a model for the language $\mathcal{L}$, built up from propositional atoms $Prop$ and the usual sub-structural connectives and constants $\vee, \wedge, \bullet, \backslash, /, \top, \bot, t$, is obtained by equipping $\mathbf{X}$ with a valuation $V : Prop \to \mathcal{F}(X)$ mapping the atoms to semilattice filters of the frame. We say that a model $(\mathbf{X}, V)$ is *topological* if $\mathbf{X}$ is an NRL-space and $V(p)$ is clopen for all $p \in Prop$. The $\curlywedge$ operation is used to interpret $\vee$ while $\otimes$ is used for $\bullet, \backslash$, and $/$, and ε the intensional truth constant, t. Below we provide conditions for $\vee, \backslash, t$ and $\bot$ as examples.

($\vee$) $x \vDash A \vee B$ iff there are $y, z \in X$ such that $y \vDash A$ and $z \vDash B$ and $y \curlywedge z \leq x$.

($\backslash$) $y \vDash A \backslash B$ iff for all $x, z \in X$, if $x \vDash A$ and $x \otimes y \leq z$, then $z \vDash B$,

(t) $x \vDash t$ iff $\varepsilon \leq x$,

($\bot$) $x \vDash \bot$ iff $x = 1$.

The condition for $\bullet$ is analogous to the one for $\vee$ and the condition for / is analogous to the one for $\backslash$. It readily follows that for any formula φ, the set $[\![\varphi]\!]$ of points satisfying φ in a given model is a filter of that model. Sequents like $\varphi \Rightarrow \psi$, which are often used to axiomatize sub-structural logics, are *true in a (topological) model* when $[\![\varphi]\!] \subseteq [\![\psi]\!]$. A sequent is valid in a frame (or NRL-space) if every model based on the frame (or space) makes it true.

From the perspective of the duality presented above, the semantics of [11, 6, 10] are naturally obtained in virtue of the facts that any algebraic model $(\mathbf{G}, \sigma)$, consisting of an $r\ell$-groupoid $\mathbf{G}$ and a valuation $\sigma : Prop \to G$, gives rise to a model $(\mathbf{X}_{\mathbf{G}}, V)$ where $V(p) = \phi(\sigma(p))$ for all $p \in Prop$. Further, we are ensured that models obtained this way are topological and that $[\![\varphi]\!]$ is a clopen filter for all $\varphi \in \mathcal{L}$. As an immediate corollary we obtain a general completeness result with respect to topological models.

Corollary 2.1. (Completeness via Duality) Every logic complete with respect to a class of $r\ell$-groupoids is complete with respect to a class of NRL-spaces.

Completeness theorems with respect to classes of frames, possibly not possessing topological structure, however, take more care. Exactly analogous to the situation in modal logic, it is not in general the case that if a sequent is valid in some NRL-space $\mathbf{X}$, then it is also valid in the underlying frame. It is therefore possible that we have topological completeness without frame completeness.

In order to obtain a result for completeness with respect to frames, and thereby recover the completeness theorems of [11, 6, 10], a notion of persistence analogous to d-persistence is needed. In [3] the notion of Π_1-persistence is introduced and can be adapted to the present setting.

Definition 2.1. We say a sequent $\varphi \Rightarrow \psi$ is Π_1-persistent if whenever $\varphi \Rightarrow \psi$ is valid in an NRL-space, then $\varphi \Rightarrow \psi$ is valid in the underlying frame of that space.

An adaptation of Theorem 4.32 in [3] yields Theorem 2.1.

Theorem 2.1. (Π_1-Persistence) Every sequent in the signature $\{\vee, \wedge, \top, \bot, \bullet, t\}$ is Π_1-persistent.

Corollary 2.2. Every logic complete with respect to a class of $r\ell$-groupoids that is axiomatized by sequents in the signature $\{\vee, \wedge, \top, \bot, \bullet, t\}$ is complete with respect to a class of frames.

The original completeness proofs in [11, 6, 10] were given for particular logics and each of their results is subsumed by Corollary 2.2. Further, the proofs of [11, 6, 10] are all canonical model style proofs and just like in the case of modal logic, it can be shown that these canonical models possess a natural NRL-space topology and are the duals of the Lindenbaum algebras of the logics in question. We can therefore make good on our promise that the canonical model style proofs of [11, 6, 10] can by analyzed in terms of algebraic completeness, duality, and a notion of persistence. In particular, we have that canonicity amounts to algebraic completeness, NRL-space duality, and Π_1-persistence.

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