A uniform semantics for connexive and paraconsistent Nelson logics

Umberto Riviecccio

Universidad Nacional de Educación a Distancia, Madrid, Spain
umberto@fsof.uned.es

Connexive logics are not only non-classical but *alternative to* (rather than just *weaker than*) classical Boolean logic, in that they validate certain classical contingencies, a prominent example being so-called *Boethius' thesis*:

$$(\varphi \to \psi) \to {\sim}(\varphi \to {\sim}\psi).$$

Connexive logics are also often *paraconsistent* (i.e. may not satisfy the classical *explosion* rule $\varphi, {\sim}\varphi \vdash \psi$), and may even be *contradictory* in Wansing's sense [8], i.e. they might have both a sentence φ and its negation ${\sim}\varphi$ as theorems.

On the other hand, propositional *paraconsistent Nelson logic* [1] is indeed a weakening of the classical calculus, usually introduced as an expansion of positive intuitionistic logic with a new De Morgan involutive negation (${\sim}$). A similar route can be taken to define certain connexive systems, notably Wansing's logic C [7]. Formally, the essential difference between paraconsistent Nelson and C is that, while in the former case we require the negation to satisfy the classical schema:

$$ {\sim}(\varphi \to \psi) \leftrightarrow (\varphi \wedge {\sim}\psi) \tag{1}$$

in the case of C – while still postulating the De Morgan and involutive laws – we reject (1) and instead impose the following version of *Boethius*:

$$ {\sim}(\varphi \to \psi) \leftrightarrow (\varphi \to {\sim}\psi). \tag{2}$$

This formal connection is exploited in the recent publication [4], in which a *twist-structure* construction is introduced for the algebraic models of C (C-*algebras*) that parallels the well-known one available for $\mathsf{N4}$-*lattices* [5], the latter being the standard algebraic models of paraconsistent Nelson logic. The two constructions diverge precisely at the point highlighted above – how to represent, in terms of the twist-structure, the negation of an implication – but neither of them is more general than the other.

It is therefore natural to consider the question whether both representations might be viewed as instances of a more general one. This is indeed possible, and even mathematically smooth. In this communication I will describe such a framework, which allows one to provide a uniform algebraic semantics encompassing Wansing's logic C and paraconsistent Nelson logic, but also more general logics, thereby providing further insight on the formal similarities shared by all these systems. Besides the aforementioned ones, it may be worthwhile to cite e.g. the three-valued logic $\mathsf{J3}$ of D'Ottaviano and da Costa [3] – already known to be a schematic extension of paraconsistent Nelson – but also, somewhat surprisingly, Cooper's *Logic of Ordinary Discourse* [2].

In our proposal a higher level of generality is achieved by dropping the requirement that the negation must satisfy the De Morgan and involutivity laws; in this way our approach also encompasses (the algebraic models of) paraconsistent logics that fail to satisfy the double negation law – such as the *quasi*-$\mathsf{N4}$-*lattices* of [6] – and may easily be employed for introducing potentially interesting new systems.

References

[1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1):231–233, 1984.

[2] William S. Cooper. The propositional logic of ordinary discourse. *Inquiry: An Interdisciplinary Journal of Philosophy*, 11(1-4):295–320, 1968.

[3] Richard L. Epstein and Itala M.L. D'Ottaviano. A paraconsistent logic: J3. In *The Semantic Foundations of Logic Volume 1: Propositional Logics*, volume 35 of *Nijhoff International Philosophy Series*, pages 263–287. Springer, Dordrecht, 1990.

[4] Davide Fazio and Sergei P. Odintsov. An algebraic investigation of the connexive logic c. *Studia Logica*, Jun 2023.

[5] S. P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76(3):385–405, 2004.

[6] Umberto Rivieccio. Quasi-N4-lattices. *Soft Computing*, 26(6):2671–2688, Mar 2022.

[7] Heinrich Wansing. Connexive modal logic. In Marcus Kracht, Maarten de Rijke, Heinrich Wansing, and Michael Zakharyaschev, editors, *Advances in Modal Logic*, pages 367–383. CSLI Publications, 2005.

[8] Heinrich Wansing. Constructive logic is connexive and contradictory. *Logic and Logical Philosophy*, pages 1–27, forthcoming.