On expressibility of satisfiability in submodels and extensions, II

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We study conditions under which certain modal operators associated with relations between first-order models are expressible in infinitary languages.

For a class K of models of a vocabulary τ, a binary relation R on K, and a sentence φ of a model-theoretic language $\mathcal{L}$ (in the sense of [1]), let $\vartheta_{K,R}(\varphi)$ express the fact that for any model $\mathfrak{A}$ in K there exists a model $\mathfrak{B}$ in K such that $\mathfrak{A}\,R\,\mathfrak{B}$ and $\mathfrak{B} \models \varphi$. Such $\vartheta_{K,R}$, treated as modal operators, lead to modal logics, which were introduced and investigated in [5] (cf. also [6], §6). One of problems arising in the study of these logics is to determine languages $\mathcal{L}'$ in which the resulting $\vartheta_{K,R}(\varphi)$ is expressible for a given φ in $\mathcal{L}$, and, in particular, to determine when it is expressible in $\mathcal{L}$ itself.

For the case when R is the submodel relation, i.e., $\mathfrak{A}\,R\,\mathfrak{B}$ means that $\mathfrak{B}$ is a submodel of $\mathfrak{A}$, and $\mathcal{L}$ is an infinitary first-order language $\mathcal{L}_{\kappa,\lambda}$ (see [1, 2]) or its second-order counterpart $\mathcal{L}^2_{\kappa,\lambda}$, this problem was investigated in [4]. It was observed there that certain $\mathcal{L}_{\kappa,\lambda}$, in particular those with inaccessible κ, are closed under the submodel operator ϑ, unlike the finitary language $\mathcal{L}_{\omega,\omega}$, which does not share this property. Some necessary and sufficient conditions of a semantic nature were given that specify sentences φ such that $\vartheta(\varphi)$ belong to $\mathcal{L}_{\omega,\omega}$.

Here we continue these studies. First, we provide a syntactic criterion for sentences φ with this property, which implies, e.g., that all Σ^0_2-sentences are such and that such sentences exist in Σ^0_n for any n. Next, we consider the converse case when R is the extension relation, i.e., $\mathfrak{A}\,R\,\mathfrak{B}$ means that $\mathfrak{A}$ is a submodel of $\mathfrak{B}$. Denoting the corresponding modal operator by ϑ^*, we observe that for φ in certain $\mathcal{L}_{\kappa,\lambda}$, in particular, with $\lambda = \omega$ or with inaccessible κ, $\vartheta^*(\varphi)$ on models of cardinality $\geqslant \kappa$ is expressible in $\mathcal{L}^2_{\kappa,\lambda}$. Then we show that for any φ in $\mathcal{L}_{\omega,\omega}$, $\vartheta^*(\varphi)$ is expressible by a Π^0_1-theory in $\mathcal{L}_{\omega,\omega}$, in striking contrast to the case of submodels as there are φ with $\vartheta(\varphi)$ non-expressible in $\mathcal{L}_{\infty,\omega}$. Furthermore, $\vartheta^*(\varphi)$ is expressible by a single sentence if and only if there exists a strongest Π^0_1-consequence of φ. This remains true for $\mathcal{L}_{\kappa,\lambda}$ with strongly compact κ and, moreover, for second-order languages $\mathcal{L}^2_{\kappa,\lambda}$ with extendible κ (for these cardinals see, e.g., [3]).

References

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