EPISTEMIC LOGIC WITH COUNTING RULES MODALITIES.

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We consider a formal epistemic system with modalities that in a sense count how many times a certain rule (or rules) must be used to derive a formula. We propose formal systems for these notions and show their completeness w.r.t. monotone neighborhood frames. We also provide estimations for the complexity of the satisfiability problem for these systems.

Keywords: Epistemic modal logic, neighborhood semantics, complexity, monotone logic

This study was inspired by an article by M.Vardi[3], which examines various epistemic systems involving non-normal logics. He proves that the satisfiability problem (SAT-problem) for a system is PSPACE-hard if it contains axiom

$$(AC)\quad \Box p \wedge \Box q \to \Box(p \wedge q).$$

If a logic contains only the following axioms

$$\begin{array}{ll} (AN_\Box) & \Box\top \\ (AM_\Box) & \Box(p \wedge q) \to \Box p \\ (AT_\Box) & \Box p \to p \\ (A4_\Box) & \Box p \to \Box\Box p \\ (A5_\Box) & \neg\Box p \to \Box\neg\Box p \end{array}$$

then the SAT-problem for it is NP-complete. Surprisingly, all these formulas do not increase the complexity of propositional logic, whose SAT problem is also NP-complete.

Axioms $(AM_\Box)$ and $(AC_\Box)$ respectively correspond to the agent's ability to use the rules $\frac{A \wedge B}{A}$ and $\frac{A,\,B}{A \wedge B}$. So, these complexity results, in some sense, correspond to the following: if an agent knows the fact $A \wedge B$, then it is not difficult for him (or her) to conclude that he (or she) knows A. However, if the agent knows A and separately knows B, it is not always easy to realize he (or she) knows $A \wedge B$.

Consider the following example. Suppose we know that Peter went to the city center and that it is raining there. It takes some effort to realize that Peter will get wet if he did not take his umbrella.

To reflect this state of affairs, we propose to consider a language with a countable number of modalities:

$$A ::= p \mid \neg A \mid A \wedge A \mid \Box_n A,$$

where $p \in PROP$ is a proposition variable and $n \in \mathbb{N}$. Intuitively we understand the formula $\Box_n A$ as "agent can deduce A from known facts using no more then n applications of rule $\frac{A,B}{A \wedge B}$".

We introduce a monotone modal logic $ENMC_\omega$ as a modal logic closed under the rules MP $(\frac{A, A\to B}{B})$, Sub $(\frac{A}{A[p/B]})$ and RE $\left(\frac{A\leftrightarrow B}{\Box A \leftrightarrow \Box B}\right)$ and that contains axioms

$(AN_{\Box_i})$, $(AM_{\Box_i})$ and the following

$$(AW_i) \qquad \Box_i p \to \Box_{i+1} p,$$
$$(AC_{ij}) \qquad \Box_i p \wedge \Box_j q \to \Box_{i+j+1}(p \wedge q).$$

Axiom AC_{ij} reflects the following: if there is a proof of φ that uses the rule $\left(\frac{A,B}{A\wedge B}\right)$ at most i times and there is a proof of ψ that uses the rule at most j times than we can prove $\varphi \wedge \psi$ using the rule at most $i + j + 1$ times.

We also consider finite counterparts of the logic $ENMC_\omega$. Let $\mathcal{ML}_n$ be the set of all modal formulas with modalities $\{\Box_0, \Box_1, \dots, \Box_n\}$ then

$$ENMC_n = ENMC_\omega \cap \mathcal{ML}_n.$$

Since the logic in question are non-normal we will need to use neighborhood semantics.

A *neighborhood frame* is a pair $F = (W, N)$, $W \neq \varnothing$, $N : W \to \mathcal{P}(\mathcal{P}(W))$.

A *neighborhood model* is a pair $M = (F, V)$, where F is a neighborhood frame, and $V : PROP \to \mathcal{P}(W)$ is a valuation. The truth relation is defined as usual, and for the boxed formula it is as follows

$$M, w \models \Box A \iff \{u \mid M, u \models A\} \in N(w).$$

The details on neighborhood semantics can be found in [1] or [2].

Theorem 1. *$ENMC_\omega$ and $ENMC_n$ are complete w.r.t. monotone neighborhood frames.*

Theorem 2. *The SAT-problem for $ENMC_n$ is NP-complete.*

Theorem 3. *The SAT-problem for $ENMC_\omega$ is in PSPACE.*

We was able to prove that $ENMC_\omega$ is in PSPACE, and it is quite possible that it is PSPACE-complete. We can offer the following interpretation. If an a priori upper bound exists on how many times an agent can use the rule $\left(\frac{A,B}{A\wedge B}\right)$, then the complexity of the epistemic logic describing the agent's knowledge is NP-complete.

It seems that the same method can be used to prove similar complexity results for similar systems, for example, those with axiom $(A4)$ and/or axiom $(A5)$, and also for multi-agent systems. In a similar way we can count the number of times the agent uses other deduction rules.

References

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