Difference–restriction algebras with operators

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Abstract

We exhibit an adjunction between a category of abstract algebras of partial functions that we call difference–restriction algebras and a category of Hausdorff étale spaces. Difference–restriction algebras are those algebras isomorphic to a collection of partial functions closed under relative complement and domain restriction; the morphisms are the homomorphisms. Our adjunction generalises the adjunction between the generalised Boolean algebras and the category of Hausdorff spaces. We define the finitary compatible completion of a difference–restriction algebra, and show that the monad induced by our adjunction yields the finitary compatible completion of any difference–restriction algebra. The adjunction restricts to a duality between the finitarily complete difference–restriction algebras and the locally compact zero-dimensional Hausdorff étale spaces, generalising the duality between generalised Boolean algebras and locally compact zero-dimensional Hausdorff spaces. We then extend these adjunction, duality, and completion results to difference–restriction algebras equipped with arbitrary additional compatibility preserving operators.

Introduction

The study of algebras of partial functions is an active area of research that investigates collections of partial functions and their interrelationships from an algebraic perspective. The partial functions are treated as abstract elements that may be combined algebraically using various natural operations such as composition, domain restriction, ‘override’, or ‘update’. In pure mathematics, algebras of partial functions arise naturally as structures such as inverse semigroups, pseudogroups, and skew lattices. In theoretical computer science, they appear in the theories of finite state transducers, computable functions, deterministic propositional dynamic logics, and separation logic. Many different selections of operations have been considered, each leading to a different category of abstract algebras (see [8, §3.2] for a guide). Recently, dualities for some of these categories have started to appear [6, 5, 7, 9, 1], opening the way for these algebras to be studied via their duals.

In [2] and [3], we initiated a project to develop a general and modular framework for producing and understanding dualities for such categories. For this we are inspired strongly by Jónsson and Tarski’s theory of Boolean algebras with operators [4] and the duality between them and descriptive general frames. Our central thesis is that in our case the appropriate base class—the analogue of Boolean algebras—must be more than just a class of *ordered* structures but must record additional *compatibility* data. This reflects the fact that the union of two partial functions is not always a function.

In [2] and [3], we investigated algebras of partial functions for a signature we believe provides the necessary order and compatibility structure. The signature has two operations: set-theoretic *relative complement* and

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a *domain restriction* operation $\triangleright$ given by: $f \triangleright g \coloneqq \{(x, y) \in X \times Y \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$. In [2], we gave and proved a finite equational axiomatisation for the class of isomorphs of such algebras of partial functions [2, Theorem 5.7]. We will refer to the algebras in this class as *difference–restriction algebras*. In [3], we gave a 'discrete' duality between the *atomic* difference–restriction algebras and a category of set quotients.

The main results of the present work are the elaboration of an adjunction between the category of difference–restriction algebras and the category of Hausdorff étale spaces (Thm. 2) and the extension of that theorem to algebras with additional operators (Thm. 6). We also show the monad induced by the adjunction gives a form of finitary completion of algebras (Thm. 3/Cor. 7(i)) and the adjunction restricts to a duality between the finitarily complete algebras and the locally complete zero-dimensional Hausdorff étale spaces (Thm. 4/Cor. 7(ii)).

Difference–restriction algebras and adjunction

An **algebra of partial functions** of the signature $\{-, \triangleright\}$ is a $\{-, \triangleright\}$-algebra whose elements are partial functions from some (common) set X to some (common) set Y, and the interpretation of $-$ is *relative complement* and the interpretation of $\triangleright$ is *domain restriction*.

A **difference–restriction algebra** is an algebra $\mathfrak{A}$ of the signature $\{-, \triangleright\}$ that is isomorphic to an algebra of partial functions. We denote by **DRA** the category whose objects are difference–restriction algebras and whose morphisms are homomorphisms of $\{-, \triangleright\}$-algebras. The operation $a \cdot b \coloneqq (a - (a - b))$ gives difference–restriction algebras a semilattice structure.

An **étale space** is a surjective local homeomorphism $\pi\colon X \twoheadrightarrow X_0$ (i.e., each $x \in X$ has an open neighbourhood U on which π restricts to a homomorphism, and $\pi(U)$ is open), and π is **Hausdorff** if X is Hausdorff. A partial function $\varphi\colon X \rightharpoonup Y$ is **continuous** if when $V \subseteq Y$ is open in Y then $\varphi^{-1}(V)$ is open in X, and φ is **proper** if whenever $V \subseteq Y$ is compact then $\varphi^{-1}(V)$ is compact.

Definition 1. We denote by **HausEt** the category whose objects are Hausdorff étale spaces $\pi\colon X \twoheadrightarrow X_0$, and where a morphism from $\pi\colon X \twoheadrightarrow X_0$ to $\rho\colon Y \twoheadrightarrow Y_0$ is a continuous and proper partial function $\varphi\colon X \rightharpoonup Y$ satisfying the following conditions:

(Q.1) φ **preserves equivalence**: if both $\varphi(x)$ and $\varphi(x')$ are defined, then $\pi(x) = \pi(x') \implies \rho(\varphi(x)) = \rho(\varphi(x'))$; thus there is an induced $\widetilde{\varphi}\colon X_0 \rightharpoonup Y_0$,

(Q.2) φ is **fibrewise injective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the restriction and co-restriction of φ induces an injective partial map $\varphi_{(x_0,y_0)}\colon \pi^{-1}(x_0) \rightharpoonup \rho^{-1}(y_0)$,

(Q.3) φ is **fibrewise surjective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the induced partial map $\varphi_{(x_0,y_0)}$ is surjective (that is, the image of $\varphi_{(x_0,y_0)}$ is the whole of $\rho^{-1}(y_0)$).

Theorem 2. *There exist adjoint functors* $F\colon \mathbf{DRA} \to \mathbf{HausEt}^{\mathrm{op}}$ *and* $G\colon \mathbf{HausEt}^{\mathrm{op}} \to \mathbf{DRA}$.

Roughly, F is 'maximal filters' and G is 'partial sections with compact image'.

Duality and completion

Two elements of a difference–restriction algebra are **compatible** if $a_1 \triangleright a_2 = a_2 \triangleright a_1$ (corresponding to partial functions agreeing on their shared domain). The algebra is **finitarily compatibly complete** provided it has joins of each finite set of pairwise-compatible elements (corresponding to being closed under finite unions of partial functions that agree wherever their domains overlap).

A **finitary compatible completion** of a difference–restriction algebra $\mathfrak{A}$ is an embedding $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $\{-, \triangleright\}$-algebras such that $\mathfrak{C}$ is a difference–restriction algebra and finitarily compatibly complete and $\iota[\mathfrak{A}]$ is finite-join dense in $\mathfrak{C}$ (i.e., each $c \in \mathfrak{C}$ is a a finite join of elements of $\iota[\mathfrak{A}]$).

Theorem 3. *For each difference–restriction algebra $\mathfrak{A}$, the homomorphism $\eta_{\mathfrak{A}}\colon \mathfrak{A} \to (G \circ F)(\mathfrak{A})$ is the finitary compatible completion of $\mathfrak{A}$, where η is the unit of the adjunction of Theorem 2.*

We write $\mathbf{C}_{\text{fin}}\mathbf{DRA}$ for the full subcategory of $\mathbf{DRA}$ consisting of the difference–restriction algebras that are finitarily compatibly complete. We write $\mathbf{Stone}^{+}\mathbf{Et}$ for the full subcategory of $\mathbf{HausEt}$ consisting of the $\pi\colon X \twoheadrightarrow X_0$ such that X is locally compact and zero-dimensional.

Proposition 4. *The adjunction restricts to a duality between $\mathbf{C}_{\text{fin}}\mathbf{DRA}$ and $\mathbf{Stone}^{+}\mathbf{Et}$.*

Adjunction for difference–restriction algebras with operators

An n-ary operation Ω on $\mathfrak{A}$ is **compatibility preserving** if whenever a_i, a_i' are compatible, for all i, we have that $\Omega(a_1, \ldots, a_n)$ and $\Omega(a_1', \ldots, a_n')$ are compatible, and Ω is an **operator** if it is *normal* ($\Omega(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$) and is *additive* (also known as *join preserving*) in each argument.

Let σ be a functional signature (disjoint from $\{-, \triangleright\}$).

The category $\mathbf{DRA}(\sigma)$ has *objects*: algebras of the signature $\{-, \triangleright\} \cup \sigma$ whose $\{-, \triangleright\}$-reduct is a difference–restriction algebra, and such that the symbols of σ are interpreted as compatibility preserving operators, and *morphisms*: homomorphisms of $(\{-, \triangleright\} \cup \sigma)$-algebras.

Let $\pi\colon X \twoheadrightarrow X_0$ be a Hausdorff étale space and R an $(n+1)$-ary relation on X. The **compatibility relation** $C \subseteq X \times X$ is given by xCy if and only if $\pi(x) = \pi(y) \implies x = y$. Then R has the **compatibility property** if given $x_1 C x_1', \ldots,\ x_n C x_n'$ and $R x_1 \ldots x_{n+1}$ and $R x_1' \ldots x_{n+1}'$, we have $x_{n+1} C x_{n+1}'$.

Given subsets $S_1, \ldots, S_n$ of X, define $\Omega_R(S_1, \ldots, S_n)$ by $\Omega_R(S_1, \ldots, S_n) := \bigcup_{x_1 \in S_1, \ldots, x_n \in S_n} \{x_{n+1} \in X \mid R x_1 \ldots x_{n+1}\}$. The relation R is **spectral** if whenever $S_1, \ldots, S_n \subseteq X$ are compact open sets, then $\Omega_R(S_1, \ldots, S_n)$ is a compact open set. The relation R is **tight** if, for each $x_1, \ldots, x_{n+1} \in X$, the condition $\forall S_1, \ldots, S_n$ compact and open $(x_1 \in S_n, \ldots, x_n \in S_n \implies x_{n+1} \in \Omega_R(S_1, \ldots, S_n))$ implies $R x_1, \ldots, x_{n+1}$.

Take a partial function $\varphi\colon X \rightharpoonup Y$ and $(n+1)$-ary relations R_X and R_Y on X and Y. Then φ satisfies the **reverse forth condition** if whenever $R_X x_1 \ldots x_{n+1}$ and $\varphi(x_1), \ldots, \varphi(x_n)$ are defined, then $\varphi(x_{n+1})$ is defined and $R_Y \varphi(x_1) \ldots \varphi(x_{n+1})$. The partial map φ satisfies the **back condition** if whenever $\varphi(x_{n+1})$ is defined and $R_Y y_1 \ldots y_n \varphi(x_{n+1})$, then there exist $x_1, \ldots, x_n \in \operatorname{dom}(\varphi)$ such that $\varphi(x_1) = y_1, \ldots, \varphi(x_n) = y_n$ and $R_X x_1 \ldots x_{n+1}$.

Definition 5. The category $\mathbf{HausEt}(\sigma)$ has *objects*: the objects of $\mathbf{HausEt}$ equipped with, for each $\Omega \in \sigma$, an $(n+1)$-ary tight spectral relation R_Ω that has the compatibility property, where n is the arity of Ω, and *morphisms*: morphisms of $\mathbf{HausEt}$ that satisfy the reverse forth condition and the back condition with respect to R_Ω, for every $\Omega \in \sigma$.

Theorem 6. *There is an adjunction $F'\colon \mathbf{DRA}(\sigma) \dashv \mathbf{HausEt}(\sigma)^{\text{op}} : G'$ that extends the adjunction $F \dashv G$ of Theorem 2 in the sense that the appropriate reducts of $F'(\mathfrak{A})$ and $G'(\pi\colon X \twoheadrightarrow X_0)$ equal $F(\mathfrak{A})$ and $G(\pi\colon X \twoheadrightarrow X_0)$, respectively.*

Corollary 7. *(i) For every algebra $\mathfrak{A}$ in $\mathbf{DRA}(\sigma)$, the embedding $\eta_{\mathfrak{A}}\colon \mathfrak{A} \hookrightarrow (G' \circ F')(\mathfrak{A})$ is the finitary compatible completion of $\mathfrak{A}$. (A finitary compatible completion in $\mathbf{DRA}(\sigma)$ should be a morphism of $\mathbf{DRA}(\sigma)$, i.e. also preserve the additional operators, as $\eta_{\mathfrak{A}}$ indeed does.)*

(ii) There is a duality between the categories $\mathbf{C}_{\text{fin}}\mathbf{DRA}(\sigma)$ and $\mathbf{Stone}^{+}\mathbf{Et}(\sigma)^{\text{op}}$.

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