Diamonds and Dominoes: Impossibility Results for Associative Modal Logics*

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In 1951, Jónsson and Tarski [13] introduced the concept of a Boolean algebra with operators (BAO), initiating the study of canonical extensions and constituting what can be construed as the basic mathematical theory of Kripke semantics for modal logic, notably predating Kripke's own work by a decade.

Fast forward to the 1990s, Kurucz et al. [18, 19] studied the decidability of Boolean algebras with a binary associative operator. Among other results, they proved that the variety of normal associative BAOs—corresponding to the least associative normal modal logic $\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$—is undecidable, arguably making it the simplest modal logic known to be undecidable.

The present research lies in continuation with the work of Kurucz and coauthors. By interpreting a domino problem, we prove that any variety of normal associative BAOs containing the complex algebra of $(\mathcal{P}(\mathbb{N}), \cup)$ is undecidable. In modal logical terms, this shows that any normal modal logic extending $\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ and sound with respect to the frame $(\mathcal{P}(\mathbb{N}), \cup)$ is undecidable. This result carries substantial implications, resolving several open questions, including the long-standing problem concerning the decidability of hyperboolean modal logic, as posed by Goranko and Vakarelov [9].

The more foundational objective of this research is to deepen our understanding of the boundary between the solvable and the unsolvable. Accordingly, this abstract also contrasts the achieved undecidability results with known decidability results. Further, in the talk, we will situate these findings within a broader landscape of undecidability including both published ([17]) and unpublished results by the author that, for brevity, are not detailed here.

Setting

Algebraically, our setting is varieties of Boolean algebras with a normal associative operator (sometimes called *Boolean semigroups*). Logically, varieties of normal associative BAOs correspond to normal extensions of the least associative normal modal logic, $\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$.

Formally, an algebra $(A, \wedge, \vee, \neg, \circ)$ is a *(normal) associative BAO* if $(A, \wedge, \vee, \neg)$ is a Boolean algebra and $\circ : A^2 \to A$ is a normal associative operator, i.e., the following holds for all $x, y, z \in A$:

- Additivity: $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- Normality: $x \circ \bot = \bot$ and $\bot \circ x = \bot$
- Associativity: $(x \circ y) \circ z = x \circ (y \circ z)$.

We will be proving our undecidability result by working dually, using that, by canonicity of associativity, the variety of associative BAOs is dually given by the class of all associative Kripke frames. Or, in other words, $\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ is the logic of associative Kripke frames, where a pair $\mathfrak{F} = (S, R)$ is an *associative (Kripke) frame* if $R \subseteq S^3$ is an associative ternary relation, i.e.,

- (Relation) associativity: $\exists v(Rvxy$ and $Rwvz)$ iff $\exists u(Ruyz$ and $Rwxu)$.

Formulas from our grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \circ \varphi,$$

where $p \in \mathbf{P}$ for $\mathbf{P}$ a set of propositional letters, are evaluated as usual in Kripke semantics. In particular, an *(associative) model* is a triple $\mathfrak{M} = (S, R, V)$ where (S, R) is an associative frame and V is a valuation

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on S, i.e., a function $V : \mathbf{P} \to \mathcal{P}(S)$. And given a model $\mathfrak{M} = (S, R, V)$, the satisfaction clauses are classical, with $\circ$ behaving as a binary diamond:

$$\mathfrak{M}, s \Vdash \varphi \circ \psi \qquad \textbf{iff} \qquad \text{there exist } s', s'' \in S \text{ such that } \mathfrak{M}, s' \Vdash \varphi;\ \mathfrak{M}, s'' \Vdash \psi;\ \text{and } Rss's''.$$

A formula φ is valid in a frame $\mathfrak{F} = (S, R)$ if for all models $\mathfrak{M} = (S, R, V)$ over $\mathfrak{F}$ and all points $s \in S$, $\mathfrak{M}, s \Vdash \varphi$. It is valid in a class of frames $\mathcal{C}$ if it is valid in each frame $\mathfrak{F} \in \mathcal{C}$, and we define the logic $\mathrm{Log}(\mathcal{C})$ of a class of frames $\mathcal{C}$ as the set of formulas valid in $\mathcal{C}$.

With this, we note that the $\{\wedge, \vee, \neg, \circ\}$-fragment of hyperboolean modal logic is $\mathrm{Log}(\mathcal{BA})$, where $\mathcal{BA}$ is the class of Boolean algebras and $\circ$ is interpreted via the relation induced by the join (or meet) of a Boolean algebra.[1] More recently, Engström and Olsson [7] have studied hyperboolean modal logic from a different perspective, introducing it as a unifying framework for propositional team logics, under the name *the Logic of Teams (LT)*.

Proof strategy and results

To establish our undecidability result, we employ a reduction from the domino (or tiling) problem, introduced by Wang [21] and shown to be undecidable by Berger [3]. This problem is formulated in terms of (Wang) tiles—unit squares with a colour on each edge—and asks whether, given a finite set of tiles $\mathcal{W}$, it is possible to cover the quadrant $\mathbb{N}^2$ so that adjacent tiles match along their shared edges. We computably construct, for each $\mathcal{W}$, a formula $\phi_{\mathcal{W}}$ such that $\phi_{\mathcal{W}}$ is valid if and only if $\mathcal{W}$ fails to tile the quadrant, thereby transferring undecidability to our setting.

To extend this result across a range of varieties, we divide this biimplication into two lemmas, each proving one direction. Due to space constraints, we only state these lemmas and show that they suffice for undecidability. For the same reason, we omit explicating the tiling formulas $\phi_{\mathcal{W}}$.

Lemma 1. *If $\mathcal{W}$ does not tile $\mathbb{N}^2$, then $\phi_{\mathcal{W}}$ is valid in the least associative normal modal logic.*

Lemma 2. *If $\phi_{\mathcal{W}}$ is valid in $(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathcal{W}$ does not tile $\mathbb{N}^2$.*

Using these two lemmas, we can derive the following theorem, stated both in algebraic and logical terms.

Theorem 3.

Let V be a variety of normal associative BAOs. If V contains the complex algebra of $(\mathcal{P}(\mathbb{N}), \cup)$, then V is undecidable.

Let L be a normal modal logic containing the associativity axiom. If $L \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then L is undecidable.

Proof. Let V be any such variety [or logic L, respectively]. By the preceding lemmas, we get that $\varphi_{\mathcal{W}}$ is valid in V [$\varphi_{\mathcal{W}} \in L$] iff $\mathcal{W}$ does not tile the quadrant. Consequently, the undecidability of the tiling problem implies the undecidability of the equational theory of V [the validity problem for L]. □

We conclude this abstract by highlighting consequences of this theorem and comparing it with known decidability results, aiming to identify traits of (un)decidable logics/varieties. First, the promised solution to the problem posed in [9].

Theorem 4.

The variety generated by complex Boolean algebras is undecidable.

Hyperboolean modal logic is undecidable.

Remark 5. As noted, [7] conceives hyperboolean modal logic as a unifying framework for propositional team logics and studies it under the name *LT*. This connection is intriguing, especially since propositional team logics are, in contrast, decidable, with the key distinction being that valuations in team semantics are principal ideals, while in *LT* they need not be. ⊣

[1] To be exact, hyperboolean modal logic is defined as the modal logic of Boolean algebras with modalities for all Boolean operators, not just the join (i.e., it's the variety generated by complex Boolean algebras). However, since undecidability in a subsignature implies undecidability in the full signature, our result extends immediately to the full hyperboolean modal logic.

Second, this solves a problem posed, in algebraic terms, by Bergman [4] and Jipsen et al. [12], and in logical terminology by the author in [14, 15].

Theorem 6.

The variety generated by complex semilattices is undecidable.

Modal (information) logic over semilattices is undecidable.

Remark 7. Interestingly, this contrasts with two decidability results, from the author's previous work. In [16], it is shown that if we expand the class of structures from semilattices to posets (so we do not require existence of all binary suprema), the resulting logic, namely modal information logic, is decidable. And in [15], it is shown that if we remove negation from our signature, leading to truthmaker semantics, the result is decidable. ⊣

Third, though not explicitly raised, we note that it follows from our result that logics recently explored by Wang and Wang [22, 23] are undecidable.

Theorem 8. *The modal logics over (distributive/modular) lattices, denoted* $\mathbb{HLSI}_L, \mathbb{HLSI}_{DL}$*, and* $\mathbb{HLSI}_{ML}$ *in [23], are all undecidable.*

Additionally, it may be worth noting that our proof yields a new proof of the following known results:

- The least associative normal modal logic, $\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$, is undecidable. (cf. [18, 19])
- Boolean Bunched Implication logic is undecidable. (cf. [5, 6, 19, 20])
- The varieties Rel and CRel of algebras isomorphic to (commutative) algebras of binary relations closed under composition and the Boolean operations $\cap, \cup, ^c$ are undecidable, obtained by combining our result with one of Jipsen [11]. (cf. [10] for Rel)

Finally, leveraging one impossibility result to prove another, we answer a somewhat different open question. Van Benthem [1, 2] shows that truthmaker semantics can be faithfully translated into modal information logic over semilattices (see also [15] for further discussion), and in [1], he poses the question whether a converse translational embedding exists. We answer this in the negative:

Theorem 9. *There is no computable function f that maps formulas φ in the language of modal information logic to a finite set of premises Φ_f and a conclusion φ_f in the language of truthmaker semantics such that, for all φ,*

φ is valid in modal information logic over semilattices iff Φ_f entails φ_f in truthmaker semantics.

Proof. We have proven that modal information logic over semilattices is undecidable, and by contrast, truthmaker consequence is not (cf. [8, 15]). □

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