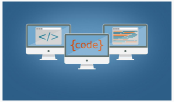
Büchi Games for the Unguarded Alternation-free μ-Calculus

Daniel Hausmann

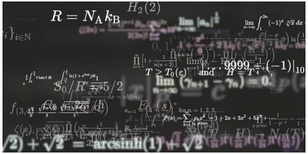
University of Liverpool, UK

TbiLLC 2025, Kutaisi, September 11, 2025

Logic in Computer Science

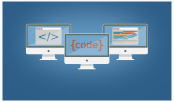


System / Model

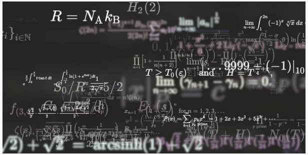


Logical Specification

Logic in Computer Science



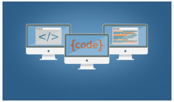
$\models$



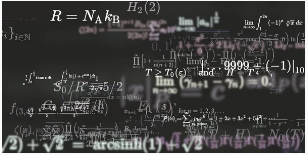
System / Model

Logical Specification

Logic in Computer Science



$\models$



System / Model

Logical Specification

Central Problems:

- Model Checking: does a given model satisfy a given specification?
- Satisfiability Checking: is there a model satisfying the specification?
- Synthesis: automatically construct system from a given specification!

Temporal Logics in Formal Methods

Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, . . .

Temporal Logics in Formal Methods

Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, ...

Formula φ: LTL / CTL, (probabilistic, graded) μ-calculus, ATL, ...

Temporal Logics in Formal Methods

Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, ...
Formula φ: LTL / CTL, (probabilistic, graded) μ-calculus, ATL, ...

Model Checking:

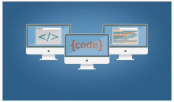
Does given model $\mathcal{M}$ satisfy given specification φ?

Temporal Logics in Formal Methods

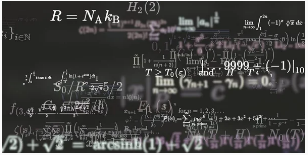
Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, . . .
Formula φ: LTL / CTL, (probabilistic, graded) μ-calculus, ATL, . . .

Model Checking:

Does given model $\mathcal{M}$ satisfy given specification φ?



$\overset{?}{\models}$



System / Model

Logical Specification

Temporal Logics in Formal Methods

Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, ...
Formula φ: LTL / CTL, (probabilistic, graded) μ-calculus, ATL, ...

Satisfiability Checking:

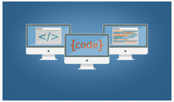
Does model satisfying given specification φ exist?

Temporal Logics in Formal Methods

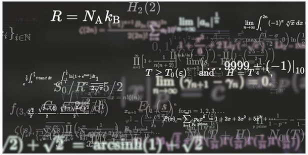
Model $\mathcal{M}$: Transition system, stochasticity, resource usage, agents, ...
Formula φ: LTL / CTL, (probabilistic, graded) μ-calculus, ATL, ...

Satisfiability Checking:

Does model satisfying given specification φ exist?



$\models$

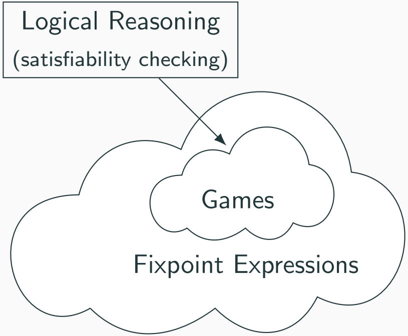


$\exists$ System / Model ?

Logical Specification

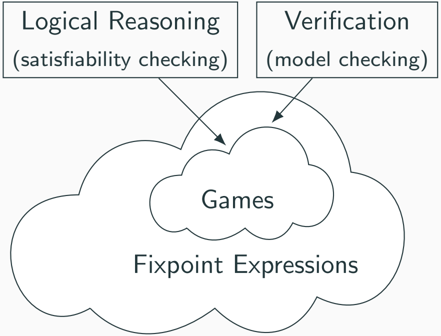
Game Reductions

Game reductions for decision problems of temporal logics:



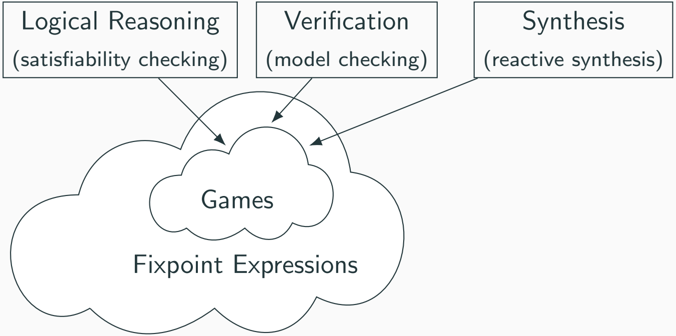
Game Reductions

Game reductions for decision problems of temporal logics:



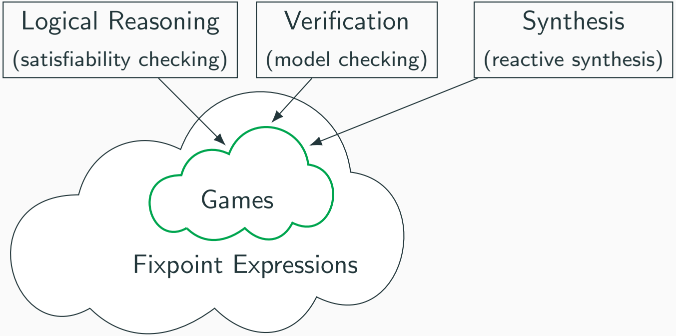
Game Reductions

Game reductions for decision problems of temporal logics:



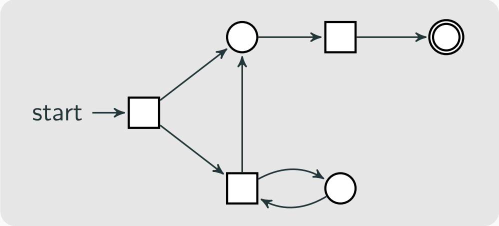
Game Reductions

Game reductions for decision problems of temporal logics:



Games on Graphs

$$G = (V_\exists, V_\forall, E, \alpha)$$

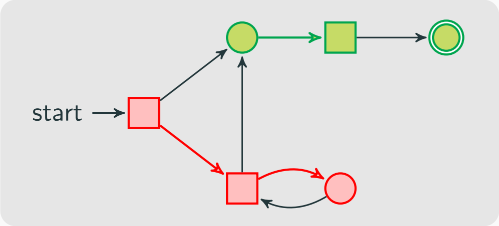


- two antagonistic players ($\exists$, $\forall$)
- (positional) $\exists$-strategy: function $s : V_\exists \to V$
- s is winning for player $\exists$ iff plays$(s) \subseteq \alpha$
- determinacy: every node is won by exactly one player

standard objectives: reachability, Büchi, co-Büchi, parity

Games on Graphs

$$G = (V_\exists, V_\forall, E, \alpha)$$



- two antagonistic players ($\exists$, $\forall$)
- (positional) $\exists$-strategy: function $s : V_\exists \to V$
- s is winning for player $\exists$ iff $\mathrm{plays}(s) \subseteq \alpha$
- determinacy: every node is won by exactly one player

standard objectives: reachability, Büchi, co-Büchi, parity

The μ-Calculus

Branching time logic, typically interpreted over transition systems

Syntax

$\varphi, \psi := p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi \mid X \mid \mu X.\,\varphi \mid \nu X.\,\varphi$ $\qquad p \in \mathrm{Prop}, X \in V$

Examples: $\mu X.\, p \vee \Box X$ $\qquad \nu X.\, \mu Y.\, (p \wedge \Diamond X) \vee \Diamond Y$

The μ-Calculus

Branching time logic, typically interpreted over transition systems

Syntax

$\varphi, \psi := p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi \mid X \mid \mu X.\,\varphi \mid \nu X.\,\varphi$ $\qquad p \in \mathsf{Prop}, X \in \mathsf{V}$

Examples: $\mu X.\, p \vee \Box X$ $\qquad \nu X.\, \mu Y.\, (p \wedge \Diamond X) \vee \Diamond Y$

Semantics

Evaluate over transition system $\mathcal{M} = (W, R, \tau)$ w.r.t. $\sigma : \mathsf{V} \to \mathcal{P}(W)$

$$[\![\Diamond\varphi]\!]_\sigma = \{v \in W \mid R(v) \cap [\![\varphi]\!]_\sigma \neq \varnothing\} \qquad [\![\mu X.\varphi]\!]_\sigma = \bigcap\{Z \subseteq W \mid [\![\varphi]\!]_\sigma(Z) \subseteq Z\}$$

$$[\![\Box\varphi]\!]_\sigma = \{v \in W \mid R(v) \subseteq [\![\varphi]\!]_\sigma\} \qquad [\![\nu X.\varphi]\!]_\sigma = \bigcup\{Z \subseteq W \mid Z \subseteq [\![\varphi]\!]_\sigma(Z)\}$$

The μ-Calculus

Branching time logic, typically interpreted over transition systems

Syntax

$\varphi, \psi := p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi \mid X \mid \mu X.\varphi \mid \nu X.\varphi$ $\qquad p \in \mathsf{Prop}, X \in \mathsf{V}$

Examples: $\mu X.\, p \vee \Box X$ $\qquad \nu X.\, \mu Y.\, (p \wedge \Diamond X) \vee \Diamond Y$

Semantics

Evaluate over transition system $\mathcal{M} = (W, R, \tau)$ w.r.t. $\sigma : \mathsf{V} \to \mathcal{P}(W)$

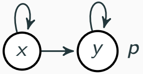
$$[\![\Diamond\varphi]\!]_\sigma = \{v \in W \mid R(v) \cap [\![\varphi]\!]_\sigma \neq \varnothing\} \qquad [\![\mu X.\varphi]\!]_\sigma = \bigcap\{Z \subseteq W \mid [\![\varphi]\!]_\sigma(Z) \subseteq Z\}$$

$$[\![\Box\varphi]\!]_\sigma = \{v \in W \mid R(v) \subseteq [\![\varphi]\!]_\sigma\} \qquad [\![\nu X.\varphi]\!]_\sigma = \bigcup\{Z \subseteq W \mid Z \subseteq [\![\varphi]\!]_\sigma(Z)\}$$

alternation depth: number $ad(\varphi)$ of alternating μ and ν operators in φ

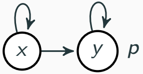
alternation-freeness: $ad(\varphi) = 1$

guardedness: every free occurrence of X in $\mu X.\varphi$ and $\nu X.\varphi$ is in scope of $\Diamond$ or $\Box$

$\mathcal{M}$: 

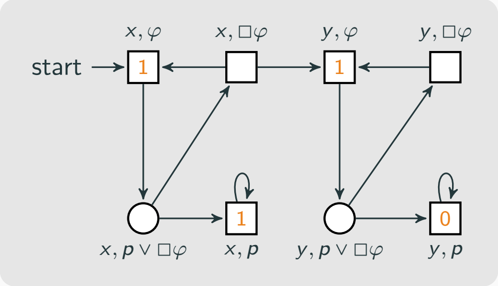
$\varphi = \mu X.\, p \vee \Box X$

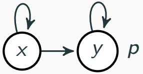
$\mathcal{M}, x \vDash \varphi$?

$\mathcal{M}$: 

$\varphi = \mu X.\, p \vee \Box X$

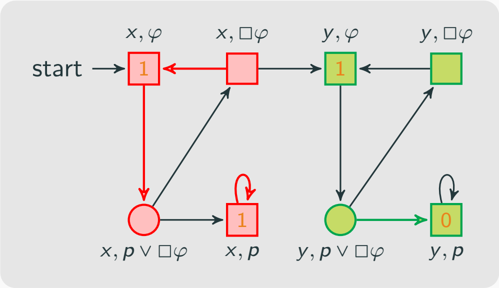
$\mathcal{M}, x \models \varphi$?



$\mathcal{M}$: 

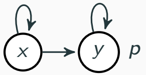
$\varphi = \mu X.\, p \vee \Box X$

$\mathcal{M}, x \models \varphi$?

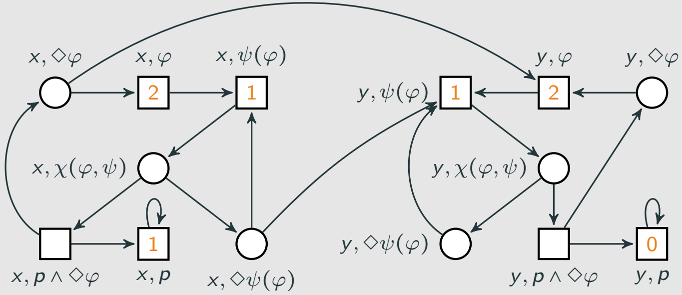


Evaluation Games, μ-Calculus

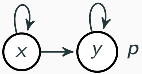
$\mathcal{M}$:



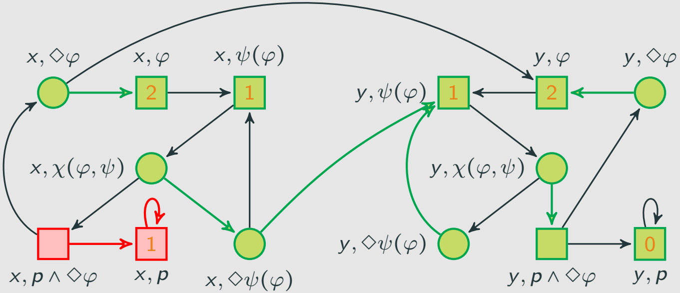
$$\varphi = \nu X.\, \overbrace{\mu Y.\, \underbrace{(p \wedge \Diamond X) \vee \Diamond Y}_{\chi(X,Y)}}^{\psi(X)}$$



Evaluation Games, μ-Calculus

$\mathcal{M}$: 

$$\varphi = \nu X.\, \overbrace{\mu Y\,.\, \underbrace{(p \wedge \Diamond X) \vee \Diamond Y}_{\chi(X,Y)}}^{\psi(X)}$$



Summary: Evaluation Games

- Evaluation games $G_{\mathcal{M},\varphi}$ are parity games with $|\mathcal{M}| \cdot |\varphi|$ nodes
- Game graph is product of model $\mathcal{M}$ and subformulae of φ
- Priorities in game correspond to alternation depths of formulae

Theorem

$x \in [[\varphi]]_{\mathcal{M}} \Leftrightarrow$ player $\exists$ wins node (x, φ) in evaluation game $G_{\mathcal{M},\varphi}$.

Corollary

Formula evaluation for the μ-calculus is in $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$ and in QP, in P for a-f fragment.

Bound on runtime $\mathcal{O}((|\mathcal{M}| \cdot |\varphi|)^{\mathsf{ad}(\varphi)})$ (with QP method: $\mathcal{O}((|\mathcal{M}| \cdot |\varphi|)^{\log(\mathsf{ad}(\varphi))})$)

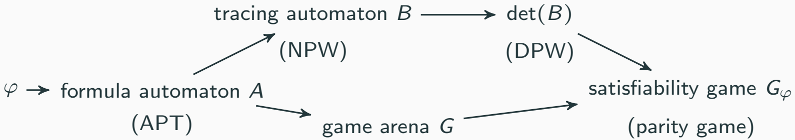
$\rightsquigarrow$ Use parity game solvers (PGSolver, Oink) to evaluate formulae

Satisfiability Games, μ-calculus

Input: μ-calculus formula φ Task: Does φ have a model?

Satisfiability Games, μ-calculus

Input: μ-calculus formula φ Task: Does φ have a model?



Theorem

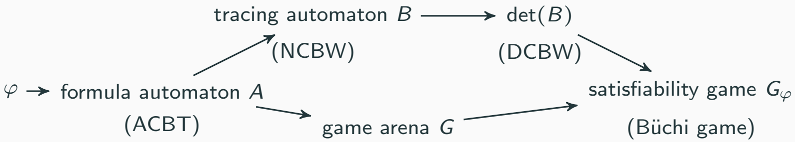
Formula φ is satisfiable if and only if player $\exists$ wins G_φ.

$|G_\varphi| \in \mathcal{O}(2^{|\varphi| \log |\varphi|})$, satisfiability problem is EXPTIME complete

☹ Determinization of Büchi automata via Safra construction (inefficient, blowup $2^{|\varphi| \log |\varphi|}$)

Satisfiability Games, alternation-free, guarded

Input: alternation-free, guarded formula φ Task: $\exists \mathcal{M}. \mathcal{M} \models \varphi$?



Theorem

Formula φ is satisfiable if and only if player $\exists$ wins G_φ.

☺ Determinization of co-Büchi automata avoids Safra construction (efficient, blowup $3^{|\varphi|}$)

$|G_\varphi| \in \mathcal{O}(3^{|\varphi|})$, satisfiability problem still EXPTIME complete

☹ Requires guardedness!

Satisfiability Games, alternation-free, possibly unguarded

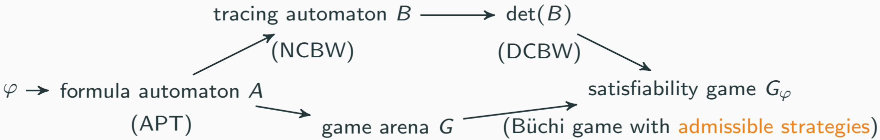
Input: alternation-free formula φ Task: $\exists \mathcal{M}. \mathcal{M} \models \varphi$?

- tracing automaton detects problematic global μ-unfoldings (as before)
- local problematic μ-unfoldings ruled out by restriction to admissible local strategies

Satisfiability Games, alternation-free, possibly unguarded

Input: alternation-free formula φ Task: $\exists \mathcal{M}.\, \mathcal{M} \models \varphi$?

- tracing automaton detects problematic global μ-unfoldings (as before)
- local problematic μ-unfoldings ruled out by restriction to admissible local strategies



☺ Determinization of co-Büchi automata avoids Safra construction (efficient, blowup $3^{|\varphi|}$)

☺ Does not require guardedness!

Satisfiability games without guardedness

Input formula χ, Fisher-Ladner closure (FL) of χ

Local strategy $s \in \mathsf{loc}$: function mapping all $\psi_0 \vee \psi_1 \in \mathsf{FL}$ to $s(\psi_0 \vee \psi_1) \in \{0,1\}$

Tracing automaton for χ with $\Sigma = \mathsf{loc} \cup \{\Diamond\varphi \mid \Diamond\varphi \in \mathsf{FL}\}$

Nondeterministic co-Büchi automaton $\mathcal{A}_\chi = (\Sigma, \mathsf{FL}, \delta, F)$ with $F = \{\varphi \in \mathsf{FL} \mid \varphi \neq \nu X.\psi\}$ and

$$\delta(\psi_0 \wedge \psi_1, s) = \{\psi_0, \psi_1\} \qquad \delta(\psi_0 \vee \psi_1, s) = \{\psi_{s(\psi_0 \vee \psi_1)}\} \qquad \delta(\eta X.\psi, s) = \{\psi[\eta X.\psi / X]\}$$

$$\delta(\Diamond\psi, \Diamond\psi) = \{\psi\} \qquad \delta(\Box\varphi, \Diamond\psi) = \{\varphi\}$$

In all other cases, put $\delta(\psi, s) = \{\psi\}$ and $\delta(\varphi, \Diamond\psi) = \emptyset$.

Determinize $\mathcal{A}_\chi$ using Miyano-Hayashi construction to $\mathcal{D}_\chi = (\Sigma, S, \Delta, B)$

Results

Local strategy s admissible at $q \in S$: s does not induce μ-trace at q in $\mathcal{D}_\chi = (\Sigma, S, \Delta, B)$

Set $H(q)$ of local strategies admissible at $q \in S$

Satisfiability game for χ

Büchi game $G_\chi = (S \times \{0,1\}, E, (S \setminus B) \times \{0,1\})$ with $2 \cdot 3^n$ nodes:

node	owner	moves to
$(q,0)$	$\exists$	$\{(\Delta(q,s^*),1) \mid s \in H(q)\}$
$(q,1)$	$\forall$	$\{(\Delta(q,\Diamond\varphi),0) \mid \Diamond\varphi \in q\}$

Theorem

Satisfiability checking for alternation-free, possibly unguarded μ-calculus formula φ in time $\mathcal{O}(3^{2|\varphi|})$. Satisfiable formulas have models of size at most $3^{|\varphi|}$.

Summary

Take-away:

- The μ-calculus: an expressive logic for specification of temporal properties over graphs
- Close relations between
  - μ-calculus and parity games
  - alternation-free μ-calculus and Büchi games
- Here: the latter correspondence does not hinge on guardedness of fixpoint variables by modal operators!