Regular Heyting algebras and free Heyting extensions of Boolean algebras

Rodrigo Nicolau Almeida – ILLC-UvA

Friday 4, 2025

Summary

1. A familiar chain of adjunctions between Heyting algebras and Boolean algebras.

Summary

1. A familiar chain of adjunctions between Heyting algebras and Boolean algebras.
2. How to study this adjunction using duality.

Summary

1. A familiar chain of adjunctions between Heyting algebras and Boolean algebras.
2. How to study this adjunction using duality.
3. Regular Heyting algebras and Inquisitive logic.

Summary

1. A familiar chain of adjunctions between Heyting algebras and Boolean algebras.
2. How to study this adjunction using duality.
3. Regular Heyting algebras and Inquisitive logic.
4. Some connections with Medvedev's logic.

Heyting Algebras and Boolean algebras

Definition
An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is called a *Heyting algebra* if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

$$a \wedge c \leq b \iff c \leq a \rightarrow b.$$

We write $\neg a := a \rightarrow 0$. It is called a *Boolean algebra* if it satisfies:

$$\forall a \in H(a \vee \neg a = 1) \text{ or } \forall a \in H(\neg\neg a = a).$$

- HA – category of Heyting algebras with Heyting algebra homomorphisms.
- BA – (full sub)category of Boolean algebras with Boolean algebra homomorphisms.

Double Negation Translation

The double negation translation of classical logic into intuitionistic logic:

Definition
Given $\phi \in \mathcal{L}_{CPC}$ we define the double negation translation into $\mathcal{L}_{IPC}$, as follows:

1. $K(p) = \neg\neg p$ and $K(\bot) = \bot$;
2. $K(\phi \wedge \psi) = K(\phi) \wedge K(\psi)$;
3. $K(\neg\phi) = \neg K(\phi)$.

Double Negation Translation

The double negation translation of classical logic into intuitionistic logic:

Definition
Given $\phi \in \mathcal{L}_{CPC}$ we define the double negation translation into $\mathcal{L}_{IPC}$, as follows:

1. $K(p) = \neg\neg p$ and $K(\bot) = \bot$;
2. $K(\phi \wedge \psi) = K(\phi) \wedge K(\psi)$;
3. $K(\neg\phi) = \neg K(\phi)$.

Theorem (Glivenko,1929)
For every formula ϕ, $\phi \in$ CPC if and only if $K(\phi) \in$ IPC.

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : $\mathsf{HA} \to \mathsf{BA}$ extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : $\mathsf{HA} \to \mathsf{BA}$ extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.
2. I : $\mathsf{BA} \to$ HA is the inclusion;

Functors, Translations and Adjunctions

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : $\mathsf{HA} \to \mathsf{BA}$ extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.
2. I : $\mathsf{BA} \to$ HA is the inclusion;
3. Reg : $\mathsf{HA} \to \mathsf{BA}$ extracts from H the BA homomorphic image $\{a : \neg\neg a = a\}$.

Functors, Translations and Adjunctions

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : **HA** $\rightarrow$ **BA** extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.
2. I : **BA** $\rightarrow$ HA is the inclusion;
3. Reg : **HA** $\rightarrow$ **BA** extracts from H the BA homomorphic image $\{a : \neg\neg a = a\}$.

Reg as a functor encapsulates the double negation translation.

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : **HA** $\to$ **BA** extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.
2. I : **BA** $\to$ HA is the inclusion;
3. Reg : **HA** $\to$ **BA** extracts from H the BA homomorphic image $\{a : \neg\neg a = a\}$.

Reg as a functor encapsulates the double negation translation.

Translations like the above correspond to *adjunctions* in the class of algebras; Reg is a right adjoint to a functor F : **BA** $\to$ **HA**.

Heyting algebras and Boolean algebras are connected in many ways:

1. Center : **HA** $\rightarrow$ **BA** extracts from H the BA subalgebra $\{a : a \vee \neg a = 1\}$.
2. I : **BA** $\rightarrow$ HA is the inclusion;
3. Reg : **HA** $\rightarrow$ **BA** extracts from H the BA homomorphic image $\{a : \neg\neg a = a\}$.

Reg as a functor encapsulates the double negation translation.

Translations like the above correspond to *adjunctions* in the class of algebras; Reg is a right adjoint to a functor F : **BA** $\rightarrow$ **HA**.

But who is F?

Free Heyting Extension

Tur and Vidal (2008) proved this functor to be fully faithful; in (A. 2023), this was studied from the point of view of a theory of translations, where a different syntactic proof was given. But the specific action of the functor was not described.

Free Heyting Extension

Tur and Vidal (2008) proved this functor to be fully faithful; in (A. 2023), this was studied from the point of view of a theory of translations, where a different syntactic proof was given. But the specific action of the functor was not described.

Our goal:

1. To use a step-by-step construction to show that this functor is connected with so called "Regular Heyting Algebras";
2. To derive from that some consequences for Inquisitive Logic and Medvedev's Logic.

Free Heyting Extension

Tur and Vidal (2008) proved this functor to be fully faithful; in (A. 2023), this was studied from the point of view of a theory of translations, where a different syntactic proof was given. But the specific action of the functor was not described.

Our goal:

1. To use a step-by-step construction to show that this functor is connected with so called "Regular Heyting Algebras";
2. To derive from that some consequences for Inquisitive Logic and Medvedev's Logic.

No shocking results: mostly categorical housekeeping, with some logical consequences.

Heyting Extensions and Esakiafication of Stone Spaces

Our main tool will be the duality between Heyting algebras and Esakia spaces:

Definition
An ordered topological space $(X, \leq, \tau)$ is said to be a Priestley space if:

1. (X, τ) is compact;
2. Whenever $x \not\leq y$ there is a clopen upwards-closed set U such that $x \in U$ and $y \notin U$;

A Priestley space is called an Esakia space if:

3. Whenever U is a clopen set, ${\downarrow}U = \{x \in X : \exists y \in U, x \leq y\}$ is clopen.

Esakia Duality

Our main tool will be the duality between Heyting algebras and Esakia spaces:

Definition
An ordered topological space $(X, \leq, \tau)$ is said to be a Priestley space if:

1. (X, τ) is compact;
2. Whenever $x \not\leq y$ there is a clopen upwards-closed set U such that $x \in U$ and $y \notin U$;

A Priestley space is called an Esakia space if:

3. Whenever U is a clopen set, $\downarrow U = \{x \in X : \exists y \in U, x \leq y\}$ is clopen.

A continuous map $p : X \to Y$ between Esakia spaces is said to be a *p-morphism* if it is order-preserving, and whenever $p(x) \leq y$, there is some $x' \geq x$ and $p(x') = y$.

Our main tool will be the duality between Heyting algebras and Esakia spaces:

Definition
An ordered topological space $(X, \leq, \tau)$ is said to be a Priestley space if:

1. (X, τ) is compact;
2. Whenever $x \not\leq y$ there is a clopen upwards-closed set U such that $x \in U$ and $y \notin U$;

A Priestley space is called an Esakia space if:

3. Whenever U is a clopen set, $\downarrow U = \{x \in X : \exists y \in U, x \leq y\}$ is clopen.

A continuous map $p : X \to Y$ between Esakia spaces is said to be a *p-morphism* if it is order-preserving, and whenever $p(x) \leq y$, there is some $x' \geq x$ and $p(x') = y$.

Theorem
There is a categorical equivalence between HA^{op} *and the category* Esa *of Esakia spaces and p-morphisms, which restricts to the Stone duality of* BA^{op} *and* Stone.

To describe $F : \mathbf{BA} \to \mathbf{HA}$, we can instead describe a dual functor $M : \mathbf{Stone} \to \mathbf{Esa}$ which is adjoint to $\mathrm{Max} : \mathbf{Esa} \to \mathbf{Stone}$ (the dual functor to Reg).

To describe $\mathrm{F} : \mathbf{BA} \longrightarrow \mathbf{HA}$, we can instead describe a dual functor $\mathrm{M} : \mathbf{Stone} \longrightarrow \mathbf{Esa}$ which is adjoint to $\mathrm{Max} : \mathbf{Esa} \longrightarrow \mathbf{Stone}$ (the dual functor to Reg).

This amounts to the following:

$$\max X \xrightarrow{\ f\ } Y$$

$$X \xrightarrow{\ \tilde{f}\ } \mathrm{M}(Y)$$

Figure 1: Adjunction Property

Example

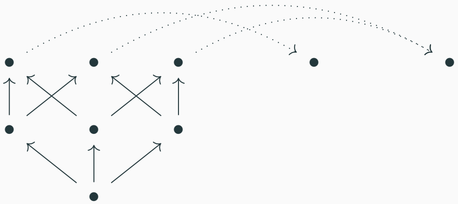

Figure 2: Example of the Problem

Bezhanishvili, Grilleti and Holliday (2019) studied a similar situation, and they provide an extension which is at least necessary.

Bezhanishvili, Grilleti and Holliday (2019) studied a similar situation, and they provide an extension which is at least necessary.

Definition
Given a Priestley space $(X, \leq)$ let $V(X)$ be the set of closed subsets with the Vietoris topology.

Bezhanishvili, Grilleti and Holliday (2019) studied a similar situation, and they provide an extension which is at least necessary.

Definition
Given a Priestley space $(X, \leq)$ let $V(X)$ be the set of closed subsets with the Vietoris topology.

Then:

Proposition
If $(X, \leq)$ is a Priestley space then:

1. $(V(X), \supseteq)$ *is an Esakia space;*
2. *If X is an Esakia space and Y is a Stone space, and $f : \max(X) \to Y$ is a continuous map, there is a unique order-preserving map $\tilde{f} : X \to V(Y)$, a p-morphism on maximal elements, which agrees on f.*

Back to the Example

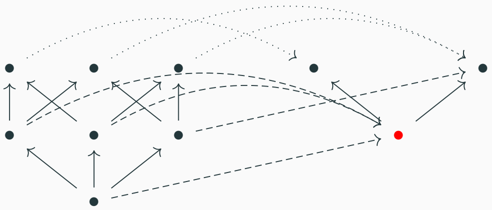

Figure 3: Back to the Example

This does not give us an adjunction because $\tilde{f}$ need not be a p-morphism. But this situation can be fixed, at a certain cost.

This does not give us an adjunction because $\tilde{f}$ need not be a p-morphism. But this situation can be fixed, at a certain cost.

Definition
Given two Priestley spaces X, Y and a continuous and order-preserving map $g : X \to Y$ between them, we say that a subset $S \subseteq X$ is *g-open* if it satisfies:

$$\forall x \in S, y \in X(x \leq y \to \exists z \in S(x \leq z \wedge g(z) = g(y))).$$

We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

This does not give us an adjunction because $\tilde{f}$ need not be a p-morphism. But this situation can be fixed, at a certain cost.

Definition
Given two Priestley spaces X, Y and a continuous and order-preserving map $g : X \to Y$ between them, we say that a subset $S \subseteq X$ is *g-open* if it satisfies:

$$\forall x \in S, y \in X(x \leq y \to \exists z \in S(x \leq z \wedge g(z) = g(y))).$$

We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

Note that if $Y = \{\bullet\}$, and g is the terminal map, $V_g(X)$ is the set of all closed and rooted subsets, which we denote by $V_r(X)$. Recall that there is a map called the *root map* $r : V_g(X) \to X$ which is a surjective order preserving map.

Note that if $Y = \{\bullet\}$, and g is the terminal map, $V_g(X)$ is the set of all closed and rooted subsets, which we denote by $V_r(X)$. Recall that there is a map called the *root map* $r : V_g(X) \to X$ which is a surjective order preserving map.

Proposition

Let Y be a Stone space and let $V_{\max}(Y) = \{C \in V_r(V(Y)) : \forall D \in C, \forall x \in D, \{x\} \in C\}$. Then $V_{\max}(Y)$ is a Priestley space, and the restriction $r : V_{\max}(Y) \to V(Y)$ is such that for any map $f : \max X \to Y$, and its unique lifting $\tilde{f} : X \to V(Y)$, there is a unique r-open $g_f : X \to V_{\max}(Y)$ making the diagram commute.

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} \dots$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V_G^r(V_{\max}(Y))$ is the inverse limit of this sequence.

Obtaining Freeness (Cont.d)

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} \dots$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V_G^r(V_{\max}(Y))$ is the inverse limit of this sequence.

$M_\infty(Y)$ is then an Esakia space, with the property that $\max(M_\infty(Y)) \cong Y$ through a natural isomorphism; moreover this assignment is functorial by using the functoriality of $V(-)$, $V_{\max}(-)$ and $V_G^r(-)$.

Obtaining Freeness (Cont.d)

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} \dots$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V_G^r(V_{\max}(Y))$ is the inverse limit of this sequence.

$M_\infty(Y)$ is then an Esakia space, with the property that $\max(M_\infty(Y)) \cong Y$ through a natural isomorphism; moreover this assignment is functorial by using the functoriality of $V(-)$, $V_{\max}(-)$ and $V_G^r(-)$.

Proposition
The functor $\mathsf{FreeM} : \mathsf{Stone} \to \mathsf{Esa}$ *assigning each Stone space* X *to* $M_\infty(X)$ *is right adjoint to* $\max : \mathsf{Esa} \to \mathsf{Stone}$.

Inquisitive Logic and Regular Heyting Algebras

Inquisitive Logic was introduced to study questions. In the work of Ciardelli, this has been reveleated to have intimate ties to intuitionistic logic; in the view of Bezhanishvili, Grilletti and Quadrellaro (2019), inquisitive logic can be seen as a non-standard logic extending intuitionistic logic.

Inquisitive Logic was introduced to study questions. In the work of Ciardelli, this has been reveleated to have intimate ties to intuitionistic logic; in the view of Bezhanishvili, Grilletti and Quadrellaro (2019), inquisitive logic can be seen as a non-standard logic extending intuitionistic logic.

In the work above, algebraic semantics are given for inquisitive logic in the form of regular Heyting algebras:

Definition
Let H be a Heyting algebra. We say that H is *regular* if $H = \langle \mathrm{Reg}(H) \rangle$. We say that an Esakia space X is regular, if its dual Heyting algebra is regular.

Regular Heyting Algebras

Given a Boolean algebra B, and its Stone space X_B, the algebra $\text{ClopUp}(V(X_B))$ has been studied as its *inquisitive extension*.

Given a Boolean algebra B, and its Stone space X_B, the algebra $\mathrm{ClopUp}(V(X_B))$ has been studied as its *inquisitive extension*.

In Grilletti and Quadrellaro (2023) a study of regular Esakia spaces was carried out. One of the questions left there is whether one can describe this class in some categorically natural way. Our main result, following from the above analysis, gives an answer:

Theorem
Given a Stone space X, $M_\infty(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are the algebras for the monad induced by this functor.

Given a Boolean algebra B, and its Stone space X_B, the algebra $\mathrm{ClopUp}(V(X_B))$ has been studied as its *inquisitive extension*.

In Grilletti and Quadrellaro (2023) a study of regular Esakia spaces was carried out. One of the questions left there is whether one can describe this class in some categorically natural way. Our main result, following from the above analysis, gives an answer:

Theorem
Given a Stone space X, $M_{\infty}(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are the algebras for the monad induced by this functor.

WARNING: Do not get confused: these are algebras on the dual side, and coalgebras on the algebraic side.

The above categorical machinery makes it easy to adapt known tools to the study of inquisitive logic:

Definition
Given $n \in \omega$ the *n-universal regular model* if the (unique) poset $(\mathcal{R}_n, \leq)$ satisfying the following:

1. $\max(P)$ contains 2^n points.
2. For each antichain $S \subseteq R_n$ where $|S| \geq 1$, there is a unique point $x \in P$ which covers S.

The above categorical machinery makes it easy to adapt known tools to the study of inquisitive logic:

Definition
Given $n \in \omega$ the *n-universal regular model* if the (unique) poset $(\mathcal{R}_n, \leq)$ satisfying the following:

1. $\max(P)$ contains 2^n points.
2. For each antichain $S \subseteq R_n$ where $|S| \geq 1$, there is a unique point $x \in P$ which covers S.

Theorem
Inquisitive logic InqL *is sound and complete with respect to the class* $\{\mathcal{R}_n : n \in \omega\}$.

N-Universal Regular Models and Categorical Considerations

There is a similar adjunction to the one here described between **Set** and **ImFinPos** the category of image-finite posets with p-morphisms.

There is a similar adjunction to the one here described between **Set** and **ImFinPos** the category of image-finite posets with p-morphisms.

The functors in that case are:

1. Con : **ImFinPos** $\to$ **Set** the connected components functor;
2. I : **Set** $\to$ **ImFinPos** the discretization functor;
3. Max : **ImFinPos** $\to$ **Set** the maximum functor;

There is a similar adjunction to the one here described between **Set** and **ImFinPos** the category of image-finite posets with p-morphisms.

The functors in that case are:

1. Con : **ImFinPos** $\rightarrow$ **Set** the connected components functor;
2. I : **Set** $\rightarrow$ **ImFinPos** the discretization functor;
3. Max : **ImFinPos** $\rightarrow$ **Set** the maximum functor;

The universal regular model as given provides the discrete analogue of the right adjoint to Max.

Definition
Let $n \in \omega$. We say that a Heyting algebra H is n-regular if H is generated from $\mathrm{Reg}(H)$ by formulas of implication depth at most n.

Then we have the following (independently obtained by Bezhanishvili and Mendler, 2025, in a different setting):

Theorem
If H is a Heyting algebra, then H is 0-regular if and only if H is a homomorphic image of an algebra $\mathrm{ClopUp}(V(X))$ for X a Stone space. In the finite case, H is a homomorphic image of the dual of $M_n \cong \mathcal{P}(n) - \{\emptyset\}$.

We finally bring the discussion to Medevedev's Logic.

Definition
Medvedev's logic Med is the logic of the frames:

$$\{V(X) : |X| = n, n \in \omega\}.$$

With some topological arguments, it is not difficult to show:

Theorem
The logic ML *is precisely the logic of all the spaces* $V(X)$ *for* X *a Stone space, and hence, the logic of all* 0*-regular Heyting algebras.*

It was shown by Grilletti and Quadrellaro (2023) that the logic of all n-regular algebras for any n is simply IPC. This opens the door to study a hierarchy of n-regular logics:

Definition
We define $\mathrm{R}_n \coloneqq \mathrm{Log}(\{H : H \text{ is } n\text{-regular}\})$.

N-Regular Logics

It was shown by Grilletti and Quadrellaro (2023) that the logic of all n-regular algebras for any n is simply IPC. This opens the door to study a hierarchy of n-regular logics:

Definition
We define $\mathsf{R}_n := \mathsf{Log}(\{H : H \text{ is } n\text{-regular}\})$.

One basic observation:

Proposition
The logic $\mathsf{R}_1 \neq \mathsf{R}_0$.

It would be interesting to know what the logics R_n yield.

Thank you!

Questions?