(Relevant) Epistemic Logic & Knowledge as Belief Based on Correct Evidence

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- $\Box Y$ $\implies$ $\Box(X \to Y)$, $\Box X$ and $\Box \neg X$ $\implies$ $\Box Y$
- K / B as entailment by an "evidential state"? Interplay K-B-E?

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Approach:

Distributive lattice logic with K and B in terms of neighbourhoods for two kinds of E (add $\to$ and $\neg$ you like).

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Distributive lattice logic with K and B in terms of neighbourhoods for two kinds of E (add $\to$ and $\neg$ you like).

Alternative? $\Box_K X := X \wedge \exists Y(\Box_U(Y \to X) \wedge \Box_r Y \wedge \Box_c Y)$. Undecidable.

Frame: $\langle S, \leqslant, N_r, N_c \rangle$ where $\langle S, \leqslant \rangle$ is a poset and, for $N \in \{N_r, N_c\}$,

$$N : \mathscr{U}(S) \to \mathscr{U}(S) \qquad N_c(X) \subseteq X$$

(that is, $s \in N(X)$ and $s \leqslant t$ only if $t \in N(X)$). We'll use $X \in N(s)$ and $s \in N(X)$ interchangeably.

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Fix an agent.

- $s \in N_r(X)$ if the agent recognises X as evidence in s (according to info in s...and all $t \geqslant s$)
- $s \in N_c(X)$ if X is correct evidence, in the context of s (...and all $t \geqslant s$: correctness is indefeasible). Correct evidence is truthful, at least.

We define three operators $B, C, K : \mathscr{U}(S) \to \mathscr{U}(S)$

$$B(X) = \{s \mid \exists Y(Y \subseteq X \ \ \& \ \ s \in N_r(Y))\}$$
$$C(X) = \{s \mid \exists Y(Y \subseteq X \ \ \& \ \ s \in N_c(Y))\}$$
$$K(X) = \{s \mid \exists Y(Y \subseteq X \ \ \& \ \ s \in N_r(Y) \cap N_c(Y))\}$$

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Example (a Gettier case). Alice and Berta are in the library. Daniel sees Alice, but not Berta, and he thinks that Alice is Camille's sister. In fact, Berta is Camille's sister.

Daniel recognises evidence “Alice is in the library and Alice is Camille's sister” for “Camille's sister is in the library”, but this evidence is not correct.
Daniel doesn't know that Camille's sister is in the library.

Completeness?

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Let $\boldsymbol{A} = \langle A, \wedge, \vee, \Box_B, \Box_C, \Box_K \rangle$ be a distributive lattice with unary operators $\Box_B, \Box_C, \Box_K$ such that

$$\Box_K a \leq \Box_B a \wedge \Box_C a \qquad \Box_C a \leq a \qquad \Box(a \wedge b) \leq \Box a \wedge \Box b$$

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Example. Complex algebra $F^+ = \langle \mathscr{U}(S), \cap, \cup, B, K \rangle$ of frame F.

Let $i \in \{0, 1\}$ and let $PF_i(\boldsymbol{A})$ be the set of pairs $\langle p, i \rangle$ where p is a prime filter on $\boldsymbol{A}$. We denote as $p_i(a)$ the set of $\langle p, i \rangle$ where $a \in p$.

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Prime filter frame $\boldsymbol{A}_+ = \left\langle \bigcup_{i \in \{0,1\}} PF_i(\boldsymbol{A}), \leqslant^{\boldsymbol{A}}, N_r^{\boldsymbol{A}}, N_c^{\boldsymbol{A}} \right\rangle$

- $\langle p, i \rangle \leqslant \langle p', i' \rangle$ iff $p \subseteq p'$ and $i = i'$
- $N_r^{\boldsymbol{A}}(s_i) = \{p_{i+1}(a) \mid \Box_B a \in s_i\} \cup \{p_0(a) \cup p_1(a) \mid \Box_K a \in s_i\}$
- $N_c^{\boldsymbol{A}}(s_i) = \{p_i(a) \mid \Box_C a \in s_i\} \cup \{p_0(a) \cup p_1(a) \mid \Box_K a \in s_i\}$

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Key Fact. $N_{rc}^{\boldsymbol{A}}(s_i) =_{df} N_r^{\boldsymbol{A}}(s_i) \cap N_c^{\boldsymbol{A}}(s_i) = \{p_0(a) \cup p_1(a) \mid \Box_K a \in s_i\}$.

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Frame Lemma.

a) $N_c^{\boldsymbol{A}}$ and $N_r^{\boldsymbol{A}}$ are monotone along $\leqslant^{\boldsymbol{A}}$.
b) $X \in N_c^{\boldsymbol{A}}(s_i)$ implies $s_i \in X$.

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Proof. Let $\widehat{a} = \bigcup_i p_i(a)$. We have

- $s_i \in \widehat{\Box_B a}$ iff $\Box_B a \in s_i$ iff $p_{i+1}(a) \in N_r^{\boldsymbol{A}}(s_i)$. Then $s_i \in B(\widehat{a})$ since $p_{i+1}(a) \subseteq \widehat{a}$. Conversely, if $\exists X \subseteq \widehat{a}$ s.t. $X \in N_r^{\boldsymbol{A}}(s_i)$, then $X = p_{i+1}(x)$ for $\Box_B x \in s_i$ or $X = p_0(x) \cup p_1(x)$ for $\Box_K x \in s_i$. In both cases $\Box_B x \in s_i$ (by $\Box_B$-mono and $\Box_K x \leq \Box_B x$). Hence, $\widehat{\Box_B a} = B(\widehat{a})$.

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- $s_i \in \widehat{\Box_K a}$ iff $\Box_K a \in s_i$ iff $p_0(a) \cup p_1(a) \in N_{rc}^{\boldsymbol{A}}(s_i)$ by Key Fact iff $\exists X \subseteq \widehat{a}$ s.t. $X \in N_{rc}^{\boldsymbol{A}}(s_i)$ by $\Box_K$-mono iff $s_i \in K(\widehat{a})$.

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- $s_i \in \widehat{\Box_C a}$ implies $p_i(a) \in N_c^{\boldsymbol{A}}(s_i)$ implies $\exists X \subseteq \widehat{a}$ s.t. $X \in N_c^{\boldsymbol{A}}(s_i)$. Conversely, $\exists X \subseteq \widehat{a}$ s.t. $X \in N_c^{\boldsymbol{A}}(s_i)$ only if $X = p_i(x)$ for $\Box_C x \in s_i$ or $X = p_0(x) \cup p_1(x)$ for $\Box_K x \in s_i$. In both cases $\Box_C x \in s_i$ (by $\Box_C$-mono and $\Box_K x \leq \Box_C x$). Hence, $\widehat{\Box_C a} = C(\widehat{a})$.

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- $s_i \in \widehat{\Box_K a}$ iff $\Box_K a \in s_i$ iff $p_0(a) \cup p_1(a) \in N_{rc}^{\boldsymbol{A}}(s_i)$ by Key Fact iff $\exists X \subseteq \widehat{a}$ s.t. $X \in N_{rc}^{\boldsymbol{A}}(s_i)$ by $\Box_K$-mono iff $s_i \in K(\widehat{a})$.

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Hence, $\widehat{\cdot}$ is hom. It is injective by the Prime Filter Theorem. □

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Stalnaker's axioms $\Box_B a \leq \Box_K \Box_B a$ (positive introspection) and $\Box_B a \leq \Box_B \Box_K a$ (strong belief) require modifications (e.g. generalised frames).

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Thank you!