On a Super-Classical System of Dialetheic Logic

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Introduction

Within this presentation, I will introduce a super-classical system of propositional logic which I refer to as “Logic of Trivalence”, or LT - the logic of *three-ness*.

“Logic of Trivalence” or LT is a conservative extension of “Strict-Tolerant Logic” with two non-classical connectives. I refer to these connectives as “alteration” - or *alt-P* for ‘alternative’ - and “mediation” - or *P mid Q* for ‘middle’.

These connectives can only be defined within a three-valued logic and the validity of their inferences requires an ST consequence relation.

Alteration and mediation also form a functionally complete set.

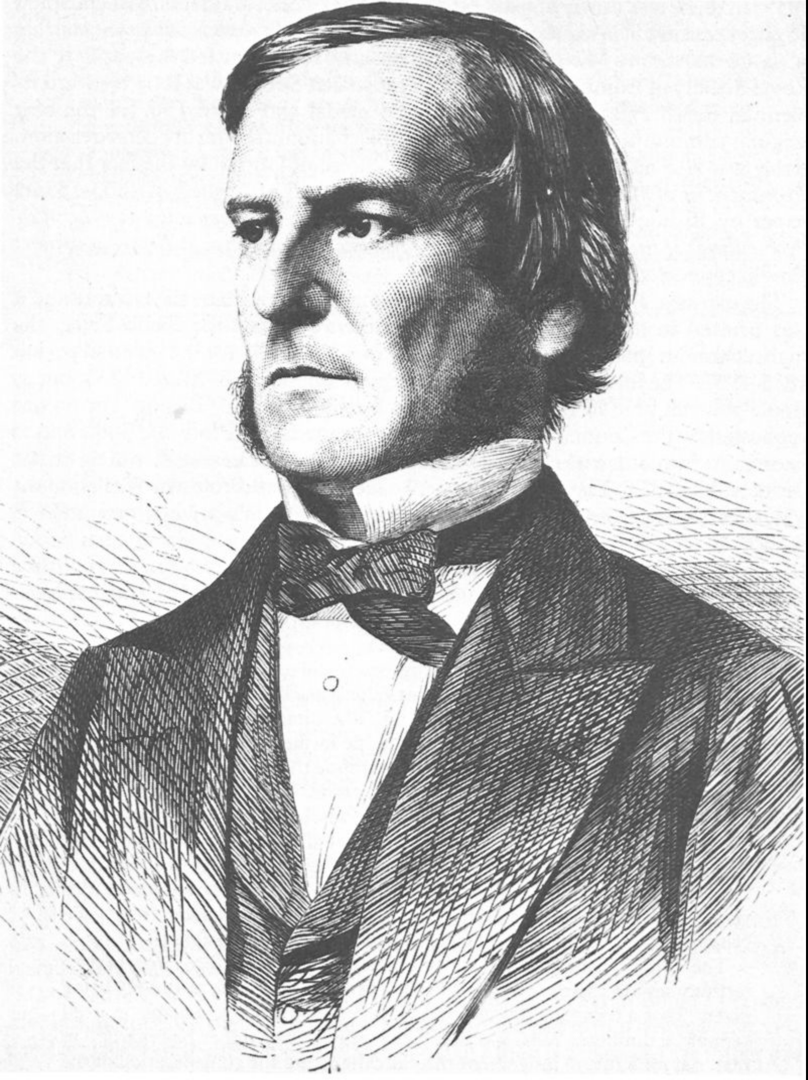
Philosophical Motivations

"Classical Logic" as a system of duality:

Dual:	**Dual:**
False 0	True 1
Conjunction $\wedge$	Disjunction $\vee$
Universal $\forall$	Particular $\exists$

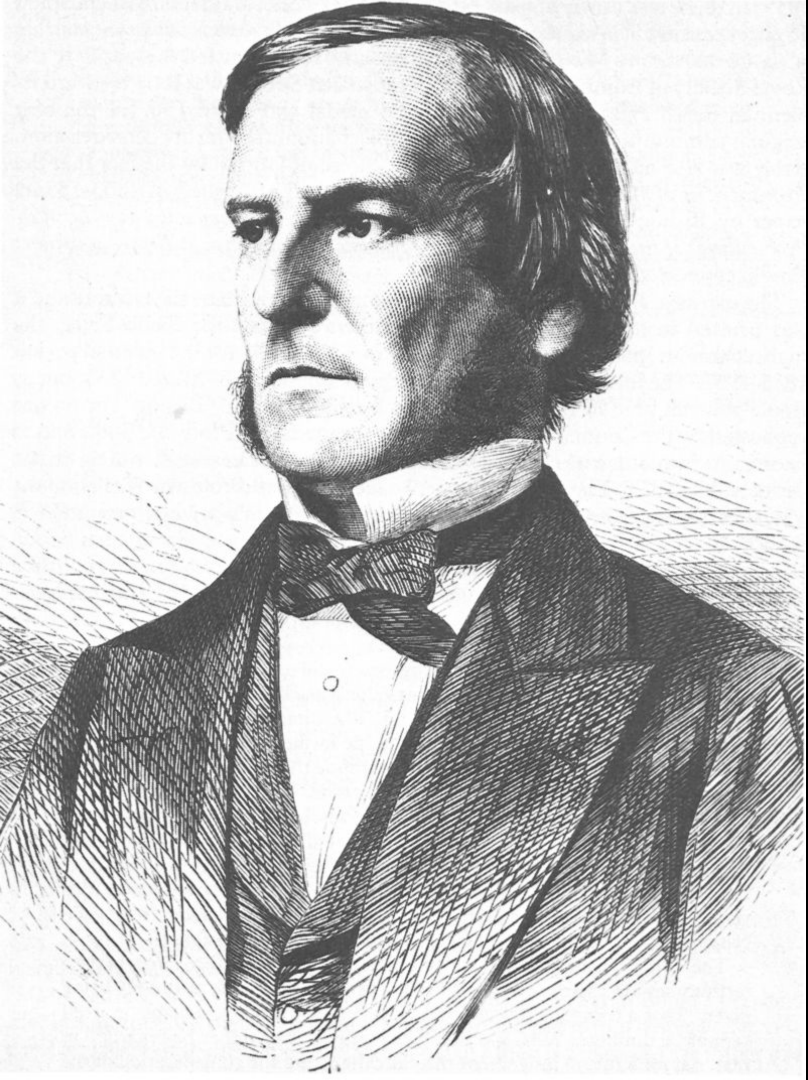
Defined in relation to one another using *negation*

What might exist *between* these duals?



"It is a consequence of the fact that the fundamental equation of thought is of the second degree, that we perform the operation of analysis and classification, by division into pairs of opposites, or, as it is technically said, by *dichotomy*."

- George Boole -



"If the equation in question had been of the third degree . . . the mental division must have been threefold in character, and we must have proceeded by a species of *trichotomy*, the real nature of which it is impossible for us, with our existing faculties, adequately to conceive, but the laws of which we might still investigate as an object of intellectual speculation."

– George Boole –

Bivalent vs. Trivalent Truth-Tables

f	1	0
1		
0		

f	1	i	0
1			
i			
0			

Such that i is interpreted as 1/2

Strict-Tolerant Logic

Negation: *not*-φ

$f\neg$	
1	**0**
i	***i***
0	**1**

Conjunction: φ *and* ψ

$f\wedge$	**1**	***i***	**0**
1	**1**	***i***	**0**
i	***i***	***i***	**0**
0	**0**	**0**	**0**

$\Gamma \vDash_{ST} \varphi$ iff: for all valuations v, if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$ or $v(\varphi) = i$

Strict-Tolerant Logic

1. ST Logic is non-transitive:

$$\top \vDash_{ST} \lambda \text{ and } \lambda \vDash_{ST} \bot \text{ but } \top \nvDash_{ST} \bot$$

2. ST Logic is inferentially classical:

$$\Gamma \vDash_{ST} \varphi \text{ iff } \Gamma \vDash_{CL} \varphi$$

3. ST Logic can be expanded with a transparent truth-predicate and self-referential sentence:

$$\neg T\langle\lambda\rangle :\equiv \lambda$$

Non-Normal Trivalent Truth-Tables

fX	1	i	0
1	i	i	i
i	i	i	i
0	i	i	i

fY	1	i	0
1	i	0	i
i	1	1	0
0	i	0	i

LT: Alteration and Mediation

Alteration: *alt*-φ

$f\sim$	
1	*i*
i	0
0	1

Mediation: φ *mid* ψ

$f\#$	1	*i*	0
1	1	1	*i*
i	1	*i*	0
0	*i*	0	0

$\Gamma \vDash_{ST} \varphi$ iff: for all valuations v, if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$ or $v(\varphi) = i$

Basic Inferences

Alteration:

$\varphi \vDash \sim\varphi$ $\quad$ $\sim\varphi \vDash \sim\sim\varphi$ $\quad$ $\sim\sim\varphi \vDash \varphi$

$\varphi \nvDash \sim\sim\varphi$ $\quad$ $\sim\sim\varphi \nvDash \sim\varphi$

$\sim\sim\sim\varphi :\equiv \varphi$

Mediation:

$\varphi \vDash \varphi \# \psi$ $\quad$ $\psi \vDash \varphi \# \psi$

$\varphi \# \psi \vDash \psi$ $\quad$ $\varphi \# \psi \vDash \varphi$

$\varphi \nvDash \psi$ $\quad$ $\psi \nvDash \varphi$

$\vDash \varphi \# \neg\varphi$ $\quad$ $\varphi \# \neg\varphi \vDash \psi$

$\neg(\varphi \# \neg\varphi) :\equiv \varphi \# \neg\varphi$

K3 and LP Consequence

Let $v(\bot) = 0$, $v(\lambda) = i$, and $v(\top) = 1$: $\top \vDash_{ST} \lambda$ and $\lambda \vDash_{ST} \bot$ but $\top \nvDash_{ST} \bot$

Alteration:

K3 Consequence: $\top \nvDash_{K3} \lambda$

$\top \nvDash_{K3} {\sim}\top$
${\sim}\top :\equiv \lambda$

${\sim}\bot \nvDash_{K3} {\sim}{\sim}\bot$
${\sim}\bot :\equiv \top$ and $\lambda :\equiv {\sim}{\sim}\bot$

${\sim}{\sim}\lambda \nvDash_{K3} \lambda$
$\top :\equiv {\sim}{\sim}\lambda$

LP Consequence: $\lambda \nvDash_{LP} \bot$

$\lambda \nvDash_{LP} {\sim}\lambda$
${\sim}\lambda :\equiv \bot$

${\sim}\top \nvDash_{LP} {\sim}{\sim}\top$
${\sim}\top :\equiv \lambda$ and $\bot :\equiv {\sim}{\sim}\top$

${\sim}{\sim}\bot \nvDash_{LP} \bot$
$\lambda :\equiv {\sim}{\sim}\bot$

Mediation:

K3 Consequence: $\top \nvDash_{K3} \lambda$

$\top \nvDash_{K3} \top \# \bot$
$\top \# \bot :\equiv \lambda$

$\top \# \lambda \nvDash_{K3} \lambda$
$\top :\equiv \top \# \lambda$

$\nvDash_{K3} \varphi \# \neg\varphi$

LP Consequence: $\lambda \nvDash_{LP} \bot$

$\lambda \nvDash_{LP} \lambda \# \bot$
$\lambda \# \bot :\equiv \bot$

$\top \# \bot \nvDash_{LP} \bot$
$\lambda :\equiv \top \# \bot$

$\varphi \# \neg\varphi \nvDash_{LP} \bot$

$\lambda :\equiv \varphi \# \neg\varphi$

Mediation is non-Trivializing

While mediation might resemble Belnap's connective of "tonk", mediation is non-trivializing.

Let $v(P) = 1$ and $v(Q) = 0$. If $v(P) = 1$ and $v(Q) = 0$, then $v(P \# Q) = i$.

If $v(P) = 1$ and $v(P \# Q) = i$, then $P \vDash_{\text{ST}} P \# Q$.

If $v(P \# Q) = i$ and $v(Q) = 0$, then $P \# Q \vDash_{\text{ST}} Q$.

However, if $v(P) = 1$ and $v(Q) = 0$, then $P \nvDash_{\text{ST}} Q$.

LT is a non-trivial extension of the classical consequence relation.

Algebraic Axioms

Law of Mediation:

$(a \wedge b) \# (a \vee b) :\equiv a \# b$

Idempotence:

$a \# a :\equiv a$

Commutativity:

$a \# b :\equiv b \# a$

DeMorgan:

$\neg(a \# b) :\equiv \neg a \# \neg b$

ST Absorption:

$a \# (a \# b) \dashv\vDash_{ST} a$

ST Associativity:

$a \# (b \# c) \dashv\vDash_{ST} (a \# b) \# c$

ST Distributivity:

$a \# (b \wedge c) \dashv\vDash_{ST} (a \# b) \wedge (a \# c)$

$a \# (b \vee c) \dashv\vDash_{ST} (a \# b) \vee (a \# c)$

Middle Element:

$a \# {\sim}a \# {\sim}{\sim}a :\equiv \lambda$

Triple Alteration:

${\sim}{\sim}{\sim}a :\equiv a$

Negation - Alteration Translation:

$\neg a :\equiv {\sim}a \# {\sim}{\sim}a$

$\neg{\sim}a :\equiv a \# {\sim}{\sim}a$

$\neg{\sim}{\sim}a :\equiv a \# {\sim}a$

Functional Completeness

These three truth-tables form a functionally complete set with respect to three-valued logic

$f!$	
1	**1**
i	**0**
0	i

$f\neg$	
1	**0**
i	i
0	**1**

$f\wedge$	**1**	i	**0**
1	**1**	i	**0**
i	i	i	**0**
0	**0**	**0**	**0**

Functional Completeness

$!\varphi :\equiv \varphi \# \sim\varphi$

$\neg\varphi :\equiv \sim\varphi \# \sim\sim\varphi$

$\varphi \wedge \psi :\equiv$

$(\sim\sim(\sim\varphi \# (\sim\varphi \# \sim\sim\varphi)) \# \sim\sim(\sim\psi \# (\sim\psi \# \sim\sim\psi)))$

$\#$

$\sim\sim(\sim\sim(\sim\sim\varphi \# (\sim\varphi \# \sim\sim\varphi)) \# \sim\sim(\sim\sim\psi \# (\sim\psi \# \sim\sim\psi)))$

Sheffer Strokes

Single Alt-Mediation: ~(φ # ψ)

f ↕	1	i	0
1	i	i	0
i	i	0	1
0	0	1	1

Double Alt-Mediation: ~~(φ # ψ)

f ⇕	1	i	0
1	0	0	1
i	0	1	i
0	1	i	i

Corresponding Quantifier

$\$x$	
$\{1,i,0\}$	i
$\{1,i\}$	1
$\{i,0\}$	0
$\{1,0\}$	i
$\{1\}$	1
$\{i\}$	i
$\{0\}$	0

Dialetheic Singularity: "for this *x*"

$$\$xAx :\equiv \forall xAx \# \exists xAx$$

$$\$xAx :\equiv \neg\$x\neg Ax \qquad \neg\$xAx :\equiv \$x\neg Ax$$

$$A(a) \vDash \$xAx \qquad \$xAx \vDash A(a)$$

$$\exists xAx \vDash \$xAx \qquad \forall xAx \vDash \$xAx$$

$$\$xAx \vDash \forall xAx \qquad \$xAx \vDash \exists xAx$$

$$\exists xAx \nvDash \forall xAx$$

Logic of Trivalence

"Logic of Trivalence" as a system of triality:

Dual:	**Self-Dual:**	**Dual:**
False 0	Both False and True *i*	True 1
Conjunction $\wedge$	Mediation #	Disjunction $\vee$
Universal $\forall$	Singular $	Particular $\exists$

. . . And they all lived happily ever after . . .

The End