Team Logics with Function Atoms
A Unified Approach

Marius Tritschler

TU Darmstadt

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First order logic with team semantics

Basic team semantics

Relational signature σ. Basic syntax of FO in NNF:

$\psi ::= \alpha \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid \exists x \psi \mid \forall x \psi$

Semantics: teams are ***sets of assignments***.

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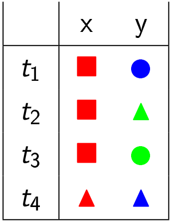
For FO_σ-literals α: $\quad \mathfrak{A}, T \Vdash \alpha \iff$ f.a. $t \in T$: $\mathfrak{A}, t \Vdash \alpha$

$T =$



$T \Vdash \mathrm{red}(x)$

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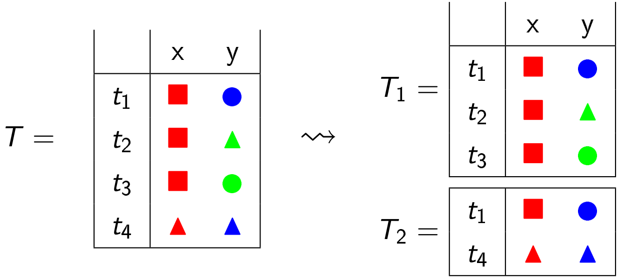
Disjunction: $\mathfrak{A}, T \Vdash \varphi_1 \vee \varphi_2 \iff$ ex. $T_1 \cup T_2 = T$ s.t. $\mathfrak{A}, T_i \Vdash \varphi_i$



$T \Vdash \text{square}(x) \vee \text{blue}(y)$

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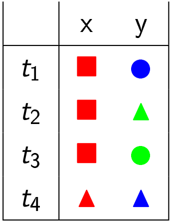
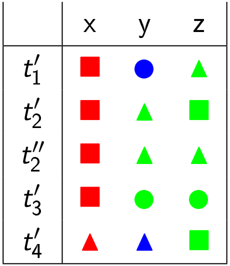
Semantics: teams are ***sets of assignments***.

$\exists$-quantification: $\quad \mathfrak{A}, T \models \exists x\varphi \quad \Longleftrightarrow \quad$ ex. set of extension(s) T' s.t. $\mathfrak{A}, T' \models \varphi$

$T =$ 

$\rightsquigarrow$

$T' =$ 

$T \models \exists z(\text{green}(z))$

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Semantics: teams are ***sets of assignments***.

$\forall$-quantification: $\quad \mathfrak{A}, T \Vdash \forall x \varphi \quad \Longleftrightarrow \quad \mathfrak{A}, T\begin{bmatrix}\mathfrak{A} \\ x\end{bmatrix} \Vdash \varphi$ (set of all extensions)

$T =$

	x	y
t_1	red ■	blue ●
t_2	red ■	green ▲
t_3	red ■	green ●
t_4	red ▲	blue ▲

$\rightsquigarrow \quad T\begin{bmatrix}\mathfrak{A} \\ x\end{bmatrix} =$

	x	y	z
t_1^1	red ■	blue ●	red ■
t_1^2	red ■	blue ●	green ■
t_1^3	red ■	blue ●	blue ■
t_1^4	red ■	blue ●	red ▲
t_1^5	red ■	blue ●	green ▲
t_1^6	red ■	blue ●	blue ▲
t_1^7	red ■	blue ●	red ●
t_1^8	red ■	blue ●	green ●
t_1^9	red ■	blue ●	blue ●
t_1^{10}	red ■	green ▲	red ■
t_1^{11}	red ■	green ▲	green ■
t_1^{12}	red ■	green ▲	blue ■
t_1^{13}	red ■	green ▲	red ▲
t_1^{14}	red ■	green ▲	green ▲

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Semantics: teams are ***sets of assignments***.

Flatness

For all $\varphi \in \mathrm{FO}_\sigma$, σ-structures $\mathfrak{A}$ and teams T of $\mathfrak{A}$:

$$\mathfrak{A}, T \models \varphi \iff \text{f.a. } t \in T\text{: } \mathfrak{A}, t \models \varphi$$

Team semantics

Team atoms

Add atoms representing team concepts:

Dependence:	$\mathfrak{A}, T \models \mathrm{dep}(\overline{x}; \overline{y})$	iff	f.a. $t, t' \in T$: $t(\overline{x}) = t'(\overline{x}) \Rightarrow t(\overline{y}) = t'(\overline{y})$.
Independence:	$\mathfrak{A}, T \models (\overline{x} \perp_{\overline{z}} \overline{y})$	iff	f.a. $\overline{a} \in \mathfrak{A}$: $T_{\overline{z} \mapsto \overline{a}}(\overline{x}, \overline{y}) = T_{\overline{z} \mapsto \overline{a}}(\overline{x}) \times T_{\overline{z} \mapsto \overline{a}}(\overline{y})$ with $T_{\overline{z} \mapsto \overline{a}} = \{t \in T \mid t(\overline{z}) = \overline{a}\}$.
Inclusion:	$\mathfrak{A}, T \models (\overline{x} \subseteq \overline{y})$	iff	$T(\overline{x}) \subseteq T(\overline{y})$.
Exclusion:	$\mathfrak{A}, T \models (\overline{x} \mid \overline{y})$	iff	$T(\overline{x}) \cap T(\overline{y}) = \emptyset$.

Corresponding team properties:

- FO(dep) is ***downward closed***: $\mathfrak{A}, T \models \varphi$ and $T' \subseteq T \implies \mathfrak{A}, T' \models \varphi$
- FO(inc) is ***union closed***: $\mathfrak{A}, T_1 \models \varphi$ and $\mathfrak{A}, T_2 \models \varphi \implies \mathfrak{A}, T_1 \cup T_2 \models \varphi$
- Empty team property $\mathfrak{A}, \emptyset \models \varphi$

Typical classification results

Existential second-order logic (ESO):
FO with ∃-quant. over relations and functions

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Existential second-order logic (ESO):
FO with $\exists$-quant. over relations and functions

All prominent team logics are fragments of ESO.

Goal: find team logic L that corresponds to a property P:

A typical classification result

For all $\varphi(R) \in$ ESO with property P there is a $\varphi^{\#}(\bar{y}) \in L$ (and vice versa) such that

$$\mathfrak{A}\begin{bmatrix} T(\bar{y}) \\ R \end{bmatrix} \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{A}, T \mathrel{\|}\!\!\models \varphi^{\#}$$

for all structures $\mathfrak{A}$ and teams T.

Team logics

Expressiveness and hierarchy

Established classification results:

Downward closure (Väänänen 2007, Kontinen, Väänänen 2009)

FO(dep) $\equiv$ downward closed ESO.
FO(dep) $\equiv$ ESO on sentences.

Full ESO (Galliani 2011)

FO(indep) $\equiv$ ESO.

Union closure (Hoelzel, Wilke 2021)

FO($\cup$ − game) $\equiv$ union closed ESO.

Translating ESO-formulae

Main ingredients I: skolemisation

How to translate ESO-formulae into team logics?

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Skolemisation (as applied in Väänänen, 2007)

For every formula $\varphi \in$ ESO there is a $\varphi^s \equiv \varphi$ of the form

$$\varphi^s = \exists f_1 \dots f_n \forall \overline{x}_1 \dots \overline{x}_n \psi$$

where $\psi \in \mathrm{FO}(f_1, \dots, f_n)$ such that f_i only appears in the form $f_i(\overline{x}_i)$ and all $\overline{x}_i, \overline{x}_j$ are pairwise disjoint.

How does this help with the translation?

- FO kernel is unproblematic due to flatness
- SO features are confined to functions

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- FO kernel is unproblematic due to flatness
- SO features are confined to (neat) functions

Translating ESO-formulae

Main ingredients II: simulating functions

Simulate functions with teams:

Swapping between functions and teams

Let $\psi \in \mathrm{FO}(f_1, \ldots, f_n)$ such that f_i only appears in the form $f_i(\overline{x}_i)$ and all $\overline{x}_i, \overline{x}_j$ are pairwise disjoint. Construct ψ^* from ψ by replacing all instances of $f_i(\overline{x}_i)$ by fresh variables z_i. Then the following are equivalent:

1. $\mathfrak{A}\left[\begin{smallmatrix} f_1^{\mathfrak{A}} \\ f_1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} f_n^{\mathfrak{A}} \\ f_n \end{smallmatrix}\right] \models \forall \overline{x}_1 \ldots \overline{x}_n \psi$.
2. $\mathfrak{A}, \mathfrak{f}_1^{\mathfrak{A}} \times \ldots \times \mathfrak{f}_n^{\mathfrak{A}} \Vdash \psi^*(\overline{x}_1, z_1, \ldots, \overline{x}_n, z_n)$.

$\mathfrak{f}_1^{\mathfrak{A}} \times \ldots \times \mathfrak{f}_n^{\mathfrak{A}}$ is the product of graphs of $f_1^{\mathfrak{A}}, \ldots, f_n^{\mathfrak{A}}$

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We have $\mathfrak{A} \models \exists f_1 \ldots f_n \forall \overline{x}_1 \ldots \overline{x}_n \psi \Longrightarrow \mathfrak{A} \Vdash \forall \overline{x}_1 \ldots \overline{x}_n \exists z_1 \ldots z_n \psi^*$

Function atoms

Dependence atoms are downward closed ***function atoms***:

$$\mathfrak{A}, T \Vdash \bigwedge_{i=1}^{n} \mathrm{dep}(\overline{x}_i; z_i) \iff \text{ex. } f_1^{\mathfrak{A}} \dots f_n^{\mathfrak{A}} \text{ s.t. } T(\overline{x}_1 z_1, \dots, \overline{x}_n z_n) \subseteq \mathfrak{f}_1^{\mathfrak{A}} \times \dots \times \mathfrak{f}_n^{\mathfrak{A}}.$$

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Introducing function atoms for other properties:

Upward closure: lax function atom

$$\mathfrak{A}, T \models \mathrm{F}(\overline{x}_1, z_1 | \dots | \overline{x}_n, z_n) \iff \text{ex. } f_1^{\mathfrak{A}} \dots f_n^{\mathfrak{A}} \text{ s.t.}$$
$$\mathfrak{f}_1^{\mathfrak{A}} \times \dots \times \mathfrak{f}_n^{\mathfrak{A}} \subseteq T(\overline{x}_1 z_1, \dots, \overline{x}_n z_n).$$

Union closure: strict function atom

$$\mathfrak{A}, T \models \mathbb{F}(\overline{x}_1, z_1 | \dots | \overline{x}_n, z_n) \iff \text{ex. } (f_{i,1}^{\mathfrak{A}} \dots f_{i,n}^{\mathfrak{A}})_{i \in I} \text{ s.t.}$$
$$T(\overline{x}_1 z_1, \dots, \overline{x}_n z_n) = \bigcup_{i \in I} \mathfrak{f}_{i,1}^{\mathfrak{A}} \times \dots \times \mathfrak{f}_{i,n}^{\mathfrak{A}}.$$

Logics with function atoms

$$\text{Let } \gamma_P := \begin{cases} \bigwedge_{i=1}^{n} \text{dep}(\overline{x}_i; z_i) & P = \text{“downward-closure”} \\ \mathrm{F}(\overline{x}_1, z_1 | \ldots | \overline{x}_n, z_n) & P = \text{“upwards-closure”} \\ \mathbb{F}(\overline{x}_1, z_1 | \ldots | \overline{x}_n, z_n) & P = \text{“union-closure”} \end{cases}.$$

$\mathfrak{A}, T \Vdash \gamma_P$ if and only if T is ***function-like*** relative to P.

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$\mathfrak{A}, T \Vdash \gamma_P$ if and only if T is ***function-like*** relative to P.

Classifications for sentences

For sentences, ESO $\equiv$ FO(dep) $\equiv$ FO(F) $\equiv$ FO($\mathbb{F}$) via

$$\mathfrak{A} \models \exists f_1 \dots f_n \forall \overline{x}_1 \dots \overline{x}_n \psi \iff \mathfrak{A} \Vdash \forall \overline{x}_1 \dots \overline{x}_n \exists z_1 \dots z_n (\gamma_P \wedge \psi^*).$$

Logics with function atoms

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Recall general classification: $\mathfrak{A}\left[\tfrac{T(\overline{y})}{R}\right] \models \varphi \iff \mathfrak{A}, T \mid\!\models \varphi^{\#}$

How to get rid of R?

Dealing with team predicate R

Normal forms & auxiliary formulae

Use ***normal forms*** corresponding to property P

$\rightsquigarrow$ Confines R to ***auxiliary*** β_P:

- Downward closed:
  $\mathfrak{A} \models \exists X(\forall \overline{w}(R\overline{w} \to X\overline{w}) \land \psi(X))$
- Upward closed:
  $\mathfrak{A} \models \exists X(\forall \overline{w}(X\overline{w} \to R\overline{w}) \land \psi(X))$
- Union closed: f.a. $\overline{a} \in R^{\mathfrak{A}}$,
  $\mathfrak{A}, \overline{a} \models \exists X(\forall \overline{w}(X\overline{w} \to R\overline{w}) \land \psi(X))$

$$\varphi$$

↓ normal form, skolemisation

$$\exists f_1 \dots f_n \forall \overline{x}_1 \dots \overline{x}_n(\beta_P \land \psi)$$

↓ Tarski/team translation

$$\forall \overline{x}_1 \dots \overline{x}_n \exists z_1 \dots z_n(\gamma_P \land \beta_P^* \land \psi^*)$$

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  $\mathfrak{A}, \overline{a} \models \exists X(\forall \overline{w}(X\overline{w} \to R\overline{w}) \wedge \psi(X))$

Last step: find $\beta_P^{\#}$ with

$\mathfrak{A}\left[{}^{T(\overline{y})}_{R}\right], T \Vdash \beta_P^{*} \iff \mathfrak{A}, T \Vdash \beta_P^{\#}$

w.r.t. function-like T.

$$\varphi$$

↓ normal form, skolemisation

$$\exists f_1 \dots f_n \forall \overline{x}_1 \dots \overline{x}_n (\beta_P \wedge \psi)$$

↓ Tarski/team translation

$$\forall \overline{x}_1 \dots \overline{x}_n \exists z_1 \dots z_n (\gamma_P \wedge \beta_P^{*} \wedge \psi^{*})$$

↓ ?

$$\forall \overline{x}_1 \dots \overline{x}_n \exists z_1 \dots z_n (\gamma_P \wedge \beta_P^{\#} \wedge \psi^{*})$$

Classification results

Classifications for sentences

For sentences, ESO $\equiv$ FO(F) $\equiv$ FO($\mathbb{F}$).

Union closure

Every union closed $\varphi(R) \in$ ESO is equivalent to a formula in FO($\mathbb{F}$), and vice versa.

Upward closure

Every upward closed $\varphi(R) \in$ ESO is equivalent to a formula in FO(F).*

(*Note that FO is not upward closed)

Convexity

Convexity

$\varphi(R) \in$ ESO is convex if for all structures $\mathfrak{A}$ and $R_l^{\mathfrak{A}} \subseteq R^{\mathfrak{A}} \subseteq R_u^{\mathfrak{A}}$,

$$\mathfrak{A}\left[\frac{R_l^{\mathfrak{A}}}{R}\right] \models \varphi \text{ and } \mathfrak{A}\left[\frac{R_u^{\mathfrak{A}}}{R}\right] \models \varphi \quad \Longrightarrow \quad \mathfrak{A}\left[\frac{R^{\mathfrak{A}}}{R}\right] \models \varphi.$$

Quasi-convexity

Quasi-convexity

$\varphi(R) \in$ ESO is quasi-convex if for all structures $\mathfrak{A}$ and $\emptyset \neq R_l^{\mathfrak{A}} \subseteq R^{\mathfrak{A}} \subseteq R_u^{\mathfrak{A}}$,

$$\mathfrak{A}\left[\tfrac{R_l^{\mathfrak{A}}}{R}\right] \models \varphi \text{ and } \mathfrak{A}\left[\tfrac{R_u^{\mathfrak{A}}}{R}\right] \models \varphi \quad \Longrightarrow \quad \mathfrak{A}\left[\tfrac{R^{\mathfrak{A}}}{R}\right] \models \varphi.$$

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Refines downward and upward closure:

- Upward and downward closed formulae are quasi-convex
- F.a. quasi-convex φ, ex. $\uparrow/\downarrow$ closed $\varphi^{\uparrow}, \varphi^{\downarrow} \in \text{ESO}$ s.t. $\varphi \equiv \varphi^{\uparrow} \wedge \varphi^{\downarrow}$.

Convex FO

- FO is quasi-convex
- dep and F atoms are quasi-convex
- $\mathrm{FO}(\mathrm{dep}, \mathrm{F})$ is not!

Neither $\vee$ nor $\exists$ preserve quasi-convexity

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Convex FO

$$\left.\begin{array}{l} T \models \varphi_1 \mathbin{\rotatebox[origin=c]{180}{V}} \varphi_2 \\ T \models \exists x \varphi \end{array}\right\} \text{iff ex. } \emptyset \neq T_l \subseteq T \subseteq T_u \text{ s.t. } \left\{\begin{array}{l} T_l, T_u \models \varphi_1 \vee \varphi_2 \\ T_l, T_u \models \exists x \varphi \end{array}\right. .$$

cvx(φ) is constructed from φ by replacing $\vee/\exists$ with $\mathbin{\rotatebox[origin=c]{180}{V}}/\exists$.

$\mathrm{FO}^{\mathrm{cvx}} := \{\mathrm{cvx}(\varphi) \mid \varphi \in \mathrm{FO}\}$.

Convex FO

Sanity check: $\varphi \equiv \text{cvx}(\varphi)$ for all $\varphi \in \text{FO}$

$\exists^{\text{c}}$ and $\forall^{\text{c}}$ preserve quasi-convexity
$\rightsquigarrow$ $\text{FO}^{\text{cvx}}(\text{dep}, \mathrm{F})$ is quasi-convex

Combine translations for upward and downward closed ESO-formulae:

Quasi-convexity

Every quasi-convex $\varphi(R) \in \text{ESO}$ is equivalent to a formula in $\text{FO}^{\text{cvx}}(\text{dep}, \mathrm{F})$.

$$\varphi(R) \equiv \left(\begin{array}{l} \forall \overline{x}^Z \overline{x}_1 \ldots \overline{x}_n \exists z_1^Z z_2^Z z_1 \ldots z_n \left(\begin{array}{l} \mathrm{dep}(\overline{x}^Z; z_1^Z) \wedge \mathrm{dep}(; z_2^Z) \\ \wedge \mathrm{dep}(\overline{x}_1; z_1) \wedge \ldots \wedge \mathrm{dep}(\overline{x}_n; z_n) \\ \wedge (\overline{y} = \overline{x}^Z \rightarrow z_1^Z = z_2^Z) \wedge (\varphi(X))^* \end{array} \right) \\ \wedge \exists \overline{x}^Y z_1^Y z_2^Y \forall \overline{x}_1 \ldots \overline{x}_n \exists z_1 \ldots z_n \overline{z}^{\exists} \left(\begin{array}{l} \mathrm{F}(\overline{x}^Y, z_1^Y; \_, z_2^Y; \overline{x}_1, z_1; \ldots; \overline{x}_n, z_n; \_, \overline{z}^{\exists}) \\ \wedge (z_1^Y = z_2^Y \rightarrow \overline{y} = \overline{x}^Y) \\ \wedge (\overline{x}^Y = \overline{z}^{\exists} \rightarrow z_1^Y = z_2^Y) \wedge (\varphi(X))^* \end{array} \right) \end{array} \right)$$

Summary

- There is a unified approach to the classification of ESO-fragments via team logics with function atoms.
- Quasi-convexity is a refinement of upward and downward closure, and there is a conservative convexification $\mathsf{FO}^{\mathsf{cvx}}$ of FO.
- The function atoms F and $\mathbb{F}$ can be used in a characterisation of the quasi-convex fragment and a novel characterisation of the union closed fragment, respectively.