Downward Closed Guarded Team Logics

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Recap: team logics

Teams are ***sets of assignments*** (think: databases).

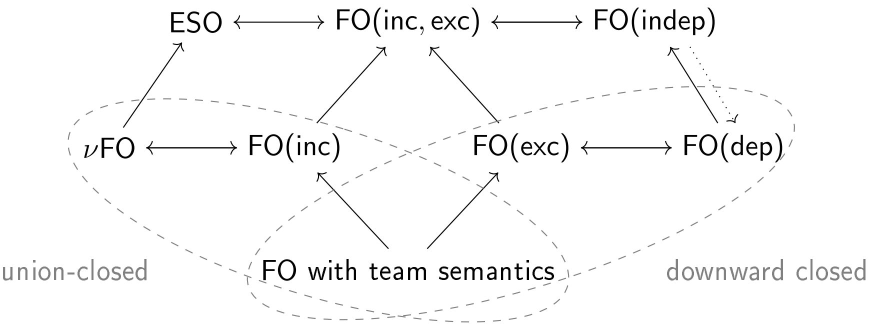
FO with team semantics is ***flat***. Expressiveness increases with the addition of ***team atoms*** for dependence (dep), inclusion (inc), exclusion (exc), ...

Downward closure: $\mathfrak{A}, T \Vdash \varphi$ and $T' \subseteq T \implies \mathfrak{A}, T' \Vdash \varphi$

Team logics L correspond to (fragments of) ***existential second order logic*** ESO: for all $\varphi(\overline{y}) \in L$, there is a $\varphi'(R) \in$ ESO s.t.

$$\mathfrak{A}, T \Vdash \varphi \iff \mathfrak{A}\begin{bmatrix} T(\overline{y}) \\ R \end{bmatrix} \models \varphi'.$$

Hierarchy of expressiveness



Expressiveness and tractability

Often, logics are studied with respect to

Expressiveness

E.g.

- Specification of structural properties (separation)
- Axiomatisability of structure classes (definability)

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Expressiveness & Tractability

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- Specification of structural properties (separation)
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E.g.

- Algorithmic model checking (satisfaction)
- Decidability (satisfiability)

Guarded logics

The guarded fragment of FO

Generalization of modal logic.

	Modal logic ML	***Guarded fragment*** GF (Andréka, van Benthem, Németi 98)
Structures:	Kripke structures (unary "colours" + binary edges)	relational structures (arbitrary hyperedges)
Quantification:	along edges	guarded by relations
Properties:	finite and tree model properties	
	decidability	
	bisimulation invariance	

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Guarded semantics

Defining feature: assignments are restricted ($\models_g$)

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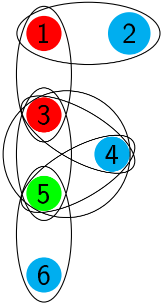

$$U^{\mathfrak{A}} = \{1, 2, 3, 4, 5, 6\}$$

$$R^{\mathfrak{A}} = \{1, 3\}, \quad B^{\mathfrak{A}} = \{2, 4, 6\}, \quad G^{\mathfrak{A}} = \{5\}$$

$$E_1^{\mathfrak{A}} = \{21, 13, 43, 53, 54, 56\}, \quad E_2^{\mathfrak{A}} = \{345\}$$

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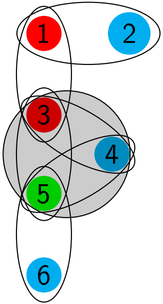

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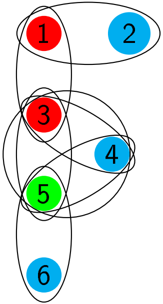

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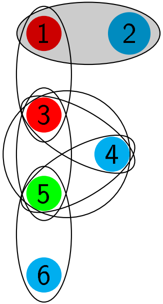

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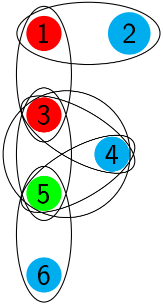

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Guarded team logics

Goal: lift concept (and properties!) of guarded logics to the team setting.

Each assignment is guarded ***individually***.

Can extend ***basic guarded team logic*** GTL with team atoms to get guarded inclusion, exclusion, dependence logics GTL(inc), GTL(exc), GTL(dep) etc.

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Guarded existential second order logic GESO has guarded SO-variables
SO-quantification does not change guards

Team logics in ESO-terms

Recall that all team logics are fragments of ESO

Prenexed GESO (pre-GESO): formulae of the form $\exists X_1 \ldots \exists X_n \psi$ with $\psi \in \mathrm{GF}$

Guarded team formulae in pre-GESO

For all $\varphi(\overline{y}) \in \mathrm{GTL}(\mathrm{inc}, \mathrm{exc})$, there is a $\varphi^{\#}(R) \in$ pre-GESO such that

$$\mathfrak{A}\begin{bmatrix} T(\overline{y}) \\ R \end{bmatrix} \models_g \varphi^{\#} \qquad \Longleftrightarrow \qquad \mathfrak{A}, T \mid\!\models_g \varphi$$

for all structures $\mathfrak{A}$ and teams T.

Proof

View Teams as sets of tuples, i.e. ***relations***.

Every clause in the evaluation of a formula can be encoded with a GF-sentence.

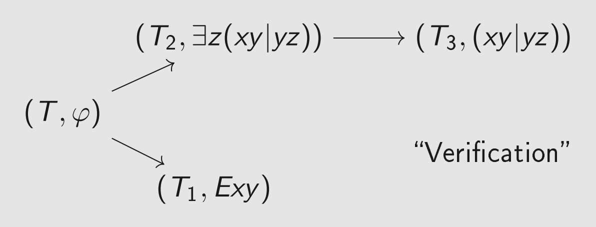
Literals	$\mathfrak{A}, T \mid\!\models \alpha$	$\forall \overline{x}(R\overline{x} \to \alpha(\overline{x}))$
Exclusion atoms	$\mathfrak{A}, T \mid\!\models (\overline{x}\mid\overline{y})$	$\forall \overline{xy}(R\overline{xy} \to \forall \overline{x}'(\neg R\overline{x}'\overline{x}))$
Disjunction	$(\psi_1 \vee \psi_2, T) \to (\psi_1, T_1)$ $(\psi_1 \vee \psi_2, T) \to (\psi_2, T_2)$ $T = T_1 \cup T_2$	$\forall \overline{x}(R\overline{x} \leftrightarrow (R_1\overline{x} \vee R_2\overline{x}))$
Ex. quantification	$(\exists x\psi, T) \longrightarrow (\psi, T')$	$\forall \overline{x}(R\overline{x} \leftrightarrow \exists y(R'\overline{xy}))$
...	...	...

Proof

by example

Let $\varphi = Exy \vee \exists z(xy|yz)$.

$\mathfrak{A}, T \Vdash_g \varphi$



$$\mathfrak{A}\begin{bmatrix} T(xy) \\ R \end{bmatrix} \models_g \exists R_1 R_2 R_3 \begin{pmatrix} \forall xy(Rxy \leftrightarrow (R_1xy \vee R_2xy)) \\ \wedge \forall xy(R_2xy \leftrightarrow \exists z(R_3xyz)) \\ \wedge \forall xyz(R_3xyz \rightarrow \forall x'(\neg R_3x'xy)) \\ \wedge \forall xy(R_1\overline{x} \rightarrow Exy) \end{pmatrix}.$$

Immediate consequences

SAT of GTL(inc, exc) reduces to SAT of GF
$\rightsquigarrow$ GTL(inc, exc) ***has finite model property, tree model property, decidability***

GTL(dep) and GESO have "infinity axioms"
$\rightsquigarrow$ GTL(dep) $\not\equiv$ GTL(exc) and GESO $\not\equiv$ pre-GESO

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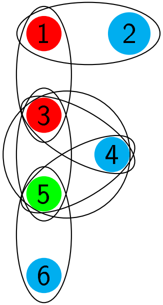
Questions:

- Guarded notions of dependence?
- GTL(exc) $\equiv$ pre-GESO on sentences?

Guarded dependence (Gdep)

Introducing ***guarded dependence*** Gdep
Detects violations of dependence only if they are guarded.



Team over xy	$\mathrm{dep}(x;y)$	$\mathrm{Gdep}(x;y)$
$\left\{\begin{array}{c}34\\34\end{array}\right\}$	✓	✓
$\left\{\begin{array}{c}35\\34\end{array}\right\}$	×	×
$\left\{\begin{array}{c}31\\34\end{array}\right\}$	×	✓

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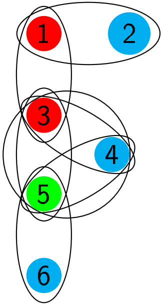

Team over xy	$\text{dep}(x; y)$	$\text{Gdep}(x; y)$
$\left\{\begin{matrix} 34 \\ 34 \end{matrix}\right\}$	✓	✓
$\left\{\begin{matrix} 35 \\ 34 \end{matrix}\right\}$	×	×
$\left\{\begin{matrix} 31 \\ 34 \end{matrix}\right\}$	×	✓

Theorem

$\text{GTL}(\text{Gdep}) \equiv \text{GTL}(\text{exc})$ *and* $\text{GTL}(\text{Gdep}) \subseteq \text{GTL}(\text{dep})$.

GTL(exc) $\neq$ pre-GESO

Subsentence decomposition

Guarded logics are restricted by ***locality***.

- Quantification corresponds to moves in the structure.
- Moves only retain information (i.e. partial assignments) if they are local.
- ***Subsentences*** correspond to global moves.
- A formula is local if there are no subsentences.

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Theorem

Every $\varphi \in \mathrm{GTL}(\mathrm{exc}, \mathrm{Gdep}, \mathrm{dep})$ *is equivalent to a positive boolean combination of local formulae and sentences.*

GTL(exc) $\neq$ pre-GESO

Isolated points

Isolated points: elements without neighbours.

Every guarded assignment containing an isolated point is ***constant***.

In naked sets (empty signature), every point is isolated.

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Theorem

Over naked sets, every sentence in $\mathrm{GTL}(\mathrm{inc}, \mathrm{dep}, \mathrm{Gdep}, \mathrm{exc})$ *is either equivalent to* $\top$, $\bot$ *or a positive boolean combination of sentences of the form* $\forall x \bigvee_{j=1}^{n} \mathrm{dep}(;x)$.

$\mathfrak{A} \models_g \forall x \bigvee_{j=1}^{n} \mathrm{dep}(;x) \iff$ "There are at most n elements in $\mathfrak{A}$".

GTL(exc) $\neq$ pre-GESO

Power of pre-GESO

pre-GESO can overcome locality in some sense:

"Colorings" allow distinguishing of non-local elements, e.g.

$$\exists X(\exists x Xx \wedge \exists y \neg Xy)$$

means "there are at least two elements."

Theorem

There are sentences in pre-GESO *that cannot be expressed in* GTL(exc, Gdep, dep).

Hybrid team logic

HTL

Introducing ***binders*** for relational variables X:

$$\mathfrak{A}, T \Vdash \downarrow_{\overline{x}} X \varphi(X) \iff \mathfrak{A}\begin{bmatrix} T(\overline{x}) \\ X \end{bmatrix}, T \Vdash \varphi(X)$$

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Can express team atoms:

$$(\overline{x} \subseteq \overline{y}) \equiv \downarrow_{\overline{y}} X(X\overline{x}), \qquad (\overline{x}|\overline{y}) \equiv \downarrow_{\overline{y}} X(\neg X\overline{x}).$$

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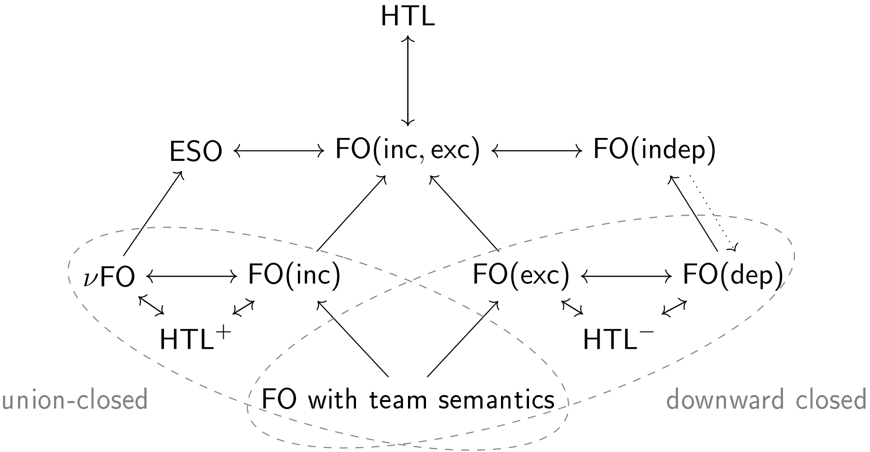
$\mathrm{HTL}^+/\mathrm{HTL}^-$: Bound relations may only appear positively/negatively

Theorem

$$\mathrm{HTL}^+ = \mathrm{FO}(\mathrm{inc}), \qquad \mathrm{HTL}^- = \mathrm{FO}(\mathrm{exc}), \qquad \mathrm{HTL} = \mathrm{FO}(\mathrm{inc}, \mathrm{exc})$$

Hybrid vs. atom-based team logics



$\mathrm{GHTL}^- \equiv$ pre-GESO for sentences

GHTL: guarded variant of HTL

Theorem

GHTL $\subseteq$ pre-GESO *and* GHTL *inherits the nice properties of* GF.

Proof: binders inherently encode teams as relations.
$\rightsquigarrow$ us the same strategy as for GTL(exc) $\subseteq$ pre-GESO

Theorem

$\mathrm{GHTL}^- \equiv$ pre-GESO *for sentences.*

Results

Theorem

GHTL $\equiv$ pre-GESO

Proof.

For all $\varphi \in$ pre-GESO, there is a $\varphi' \in$ GHTL^{-} such that

$$\mathfrak{A}\left[{}^{T(\overline{x})}_{R}\right] \models_g \varphi \iff \mathfrak{A}\left[{}^{T(\overline{x})}_{R}\right] \Vdash_g \varphi' \iff \mathfrak{A}, T \Vdash_g {\downarrow_{\overline{x}}} R(\varphi').$$

□

Theorem

GHTL^{-} *is the downward-closed fragment of* pre-GESO.

Proof.

Similar as before, with modifications to φ' so that R occurs only negatively. □

Conclusion

- Guarded logics have restricted expressive power, but nice model-theoretic properties. This transfers to guarded team logics like GTL(exc).
- Hybrid logics overcome some of the restrictions while keeping the properties.
- The expressive hierarchy is significantly more varied than in the non-guarded case.