Degree of Kripke Incompleteness in Tense Logics

Qian Chen[1,2]

[1] The Tsinghua-UvA Joint Research Centre for Logic,
Department of Philosophy, Tsinghua University, Beijing, China

[2] ILLC, University of Amsterdam, Amsterdam, Netherlands.

TbiLLC 2025, Tskaltubo, Georgia

Outline

1. Degree of Kripke-incompleteness
2. Dichotomy Theorem for NExt($S4_t$)

Outline

1. Degree of Kripke-incompleteness

2. Dichotomy Theorem for NExt($S4_t$)

Kripke-completeness of modal logic

A Kripke-frame is a pair $\mathfrak{F} = (X, R)$ where X is a non-empty set and R a binary relation on X

A normal modal logic is a set of formulas $L \supseteq \mathsf{CPC}$ closed under (MP), (Nec) and (Sub)
Let K be the least normal modal logic and NExt(K) denote the lattice of all normal modal logic

Kripke-completeness of modal logic

A Kripke-frame is a pair $\mathfrak{F} = (X, R)$ where X is a non-empty set and R a binary relation on X

A normal modal logic is a set of formulas $L \supseteq \mathsf{CPC}$ closed under (MP), (Nec) and (Sub)
Let K be the least normal modal logic and NExt(K) denote the lattice of all normal modal logic

For each class $\mathcal{K}$ of frames, we write $\mathsf{Log}(\mathcal{K})$ for the set of formulas

$$\{\varphi : \mathcal{K} \models \varphi\}$$

For each modal logic L, we write $\mathsf{Fr}(L)$ for the class of frames

$$\{\mathfrak{F} : \mathfrak{F} \models L\}$$

A modal logic L is Kripke-complete if $L = \mathsf{Log}(\mathsf{Fr}(L))$

Kripke-completeness

Kripke-completeness of modal logics has been extensively studied since 1960s.

Kripke-completeness

Kripke-completeness of modal logics has been extensively studied since 1960s.

[Thomason, 1972] established the existence of Kripke-incomplete tense logics

[Fine, 1974] and [Van Benthem, 1978] gave examples of Kripke-incomplete modal logics

Kripke-completeness

Kripke-completeness of modal logics has been extensively studied since 1960s.

[Thomason, 1972] established the existence of Kripke-incomplete tense logics

[Fine, 1974] and [Van Benthem, 1978] gave examples of Kripke-incomplete modal logics

One step further? Where are Kripke-incomplete logics located in the lattice NExt(K)?

Kripke-completeness

Kripke-completeness of modal logics has been extensively studied since 1960s.

[Thomason, 1972] established the existence of Kripke-incomplete tense logics

[Fine, 1974] and [Van Benthem, 1978] gave examples of Kripke-incomplete modal logics

One step further? Where are Kripke-incomplete logics located in the lattice NExt(K)?

Degree of Kripke-incompleteness!

Degree of Kripke-incompleteness

Fine introduced the degree of Kripke-incompleteness of logics [Fine, 1974]

Degree of Kripke-incompleteness

Fine introduced the degree of Kripke-incompleteness of logics [Fine, 1974]

Let $\mathcal{L}$ be a lattice of logics and $\equiv_{\mathsf{Fr}}$ the equivalence relation on $\mathcal{L}$ such that

$$L_1 \equiv_{\mathsf{Fr}} L_2 \text{ iff } L_1 \text{ shares the same class of frames as } L_2\text{, i.e., } \mathsf{Fr}(L_1) = \mathsf{Fr}(L_2).$$

For each $L \in \mathcal{L}$, let

$$[L]_{\equiv_{\mathsf{Fr}}} := \{L' \in \mathcal{L} : \mathsf{Fr}(L) = \mathsf{Fr}(L')\}$$

The *degree of Kripke-incompleteness* $\deg_{\mathcal{L}}(L)$ *of L in $\mathcal{L}$* is defined to be the cardinality of $[L]_{\equiv_{\mathsf{Fr}}}$

L is said to be *strictly Kripke-complete in $\mathcal{L}$* if $\deg_{\mathcal{L}}(L) = 1$

Blok's dichotomy theorem on Kripke-incompleteness

One of the most important result on Kripke-incompleteness in NExt(K) [Blok, 1978]:

- every modal logic $L \in \mathrm{NExt}(\mathrm{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$

Blok's dichotomy theorem on Kripke-incompleteness

One of the most important result on Kripke-incompleteness in NExt(K) [Blok, 1978]:

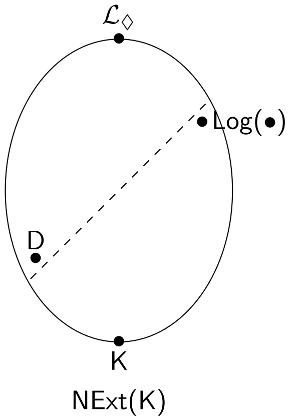
- every modal logic $L \in \mathrm{NExt}(\mathrm{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$
- union-splittings in NExt(K) are exactly the strictly Kripke-complete logics

Splittings

Let $\mathcal{L}$ be a lattice of logics and $L_1, L_2 \in \mathcal{L}$. Then $\langle L_1, L_2 \rangle$ is called a splitting pair in $\mathcal{L}$ if,

$$\text{for all } L \in \mathcal{L}, \text{ exactly one of } L \subseteq L_1 \text{ and } L \supseteq L_2 \text{ holds}$$

We say L_1 splits the lattice $\mathcal{L}$ and we call L_2 the splitting of $\mathcal{L}$ by L_1 and denote it by $\mathcal{L}/L_1$ (if $\mathcal{L} = \mathrm{NExt}(L_0)$, we also write L_0/L_1)



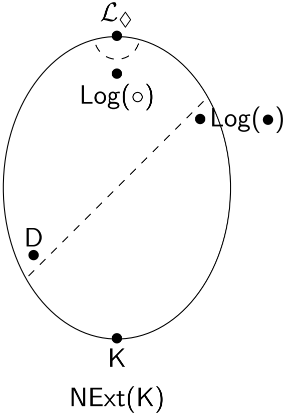
Splittings

Let $\mathcal{L}$ be a lattice of logics and $L_1, L_2 \in \mathcal{L}$. Then $\langle L_1, L_2 \rangle$ is called a splitting pair in $\mathcal{L}$ if,

$$\text{for all } L \in \mathcal{L}, \text{ exactly one of } L \subseteq L_1 \text{ and } L \supseteq L_2 \text{ holds}$$

We say L_1 splits the lattice $\mathcal{L}$ and we call L_2 the splitting of $\mathcal{L}$ by L_1 and denote it by $\mathcal{L}/L_1$ (if $\mathcal{L} = \mathrm{NExt}(L_0)$, we also write L_0/L_1)

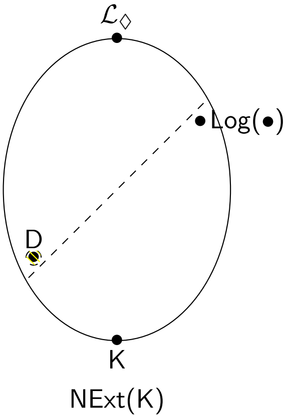
L is called a *union-splitting* in $\mathcal{L}$ if $L = \bigoplus_{i \in I} L_i$ for some family $\{L_i : i \in I\}$ of splittings
L is called a *iterated splitting* in $\mathcal{L}$ if $L = \mathcal{L}/L_1/L_2/\cdots/L_n$ for some $L_1, L_2, \cdots, L_n$ such that L is well-defined
(specially, if $\mathcal{L} = \mathrm{NExt}(L_0)$, L_0 is also an iterated splitting)



Blok's theorem on Kripke-incompleteness

Blok's theorem on Kripke-incompleteness in NExt(K) [Blok, 1978]:

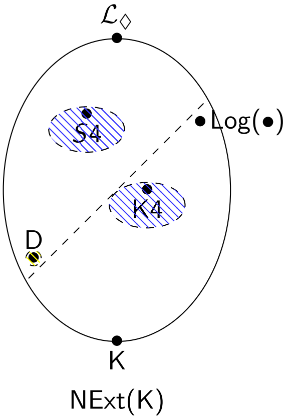
- every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$
- union-splittings in NExt(K) are exactly the consistent strictly Kripke-complete logics
- iterated splittings in NExt(K) are exactly the strictly Kripke-complete logics



Blok's theorem on Kripke-incompleteness

Blok's theorem on Kripke-incompleteness in NExt(K) [Blok, 1978]:

- every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$
- union-splittings in NExt(K) are exactly the consistent strictly Kripke-complete logics
- iterated splittings in NExt(K) are exactly the strictly Kripke-complete logics



More on the Degree of Incompleteness

Study the degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{K}) \Longrightarrow$ Study $\equiv_{\mathsf{Fr}}$ in $\mathsf{NExt}(\mathsf{K})$

More on the Degree of Incompleteness

Study the degree of Kripke-incompleteness in NExt(K) $\Longrightarrow$ Study $\equiv_{\mathsf{Fr}}$ in NExt(K)

What if we replace Fr with some other class $\mathcal{C}$ of structures?

More on the Degree of Incompleteness

Study the degree of Kripke-incompleteness in NExt(K) $\Longrightarrow$ Study $\equiv_{\mathsf{Fr}}$ in NExt(K)

What if we replace Fr with some other class $\mathcal{C}$ of structures?

- Modal algebras MA: every normal modal logic is strictly MA-complete
- Neighborhood frames NF: [Chagrova, 1998], [Dziobiak, 1978] and [Litak, 2004] ...

More on the Degree of Incompleteness

Study the degree of Kripke-incompleteness in NExt(K) $\Longrightarrow$ Study $\equiv_{\mathsf{Fr}}$ in NExt(K)

What if we replace Fr with some other class $\mathcal{C}$ of structures?

- Modal algebras MA: every normal modal logic is strictly MA-complete
- Neighborhood frames NF: [Chagrova, 1998], [Dziobiak, 1978] and [Litak, 2004] ...

Degree of incompleteness in different lattices of logics, instead of NExt(K)

- A longstanding open problem: Does Blok's dichotomy theorem hold for K4, S4, or IPC?
- [Fornasiere and Moraschini, 2024]: Degrees of Kripke-incompleteness of implicative logics:
  - the trichotomy theorem: the degree is one of 1, $\aleph_0$ and $2^{\aleph_0}$

More on the Degree of Incompleteness

We could also do some combination:

- [Bezhanishvili et al., 2025]: Anti-dichotomy theorem of the degree of FMP for K4, S4, or IPC: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists L of degree of FMP κ

More on the Degree of Incompleteness

We could also do some combination:

- [Bezhanishvili et al., 2025]: Anti-dichotomy theorem of the degree of FMP for K4, S4, or IPC: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists L of degree of FMP κ
- [Dziobiak, 1978]: dichotomy theorem for the degree of NF-incompleteness in the lattice $\text{Ext}(\text{D} \oplus (\Box^n p \to \Box^{n+1} p))$ for all $n \in \omega$

More on the Degree of Incompleteness

We could also do some combination:

- [Bezhanishvili et al., 2025]: Anti-dichotomy theorem of the degree of FMP for K4, S4, or IPC: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists L of degree of FMP κ
- [Dziobiak, 1978]: dichotomy theorem for the degree of NF-incompleteness in the lattice $\mathrm{Ext}(\mathsf{D} \oplus (\Box^n p \to \Box^{n+1} p))$ for all $n \in \omega$
- [Fornasiere and Moraschini, 2024]: Degrees of Kripke-incompleteness of implicative logics:
    - the trichotomy theorem: the degree is one of 1, $\aleph_0$ and $2^{\aleph_0}$

More on the Degree of Incompleteness

We could also do some combination:

- [Bezhanishvili et al., 2025]: Anti-dichotomy theorem of the degree of FMP for K4, S4, or IPC: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists L of degree of FMP κ
- [Dziobiak, 1978]: dichotomy theorem for the degree of NF-incompleteness in the lattice $\mathrm{Ext}(\mathrm{D} \oplus (\Box^n p \to \Box^{n+1} p))$ for all $n \in \omega$
- [Fornasiere and Moraschini, 2024]: Degrees of Kripke-incompleteness of implicative logics:
  - the trichotomy theorem: the degree is one of 1, $\aleph_0$ and $2^{\aleph_0}$

In this talk, we focus on the degree of Kripke-incompleteness of *tense logics*

Tense Logics

A tense logic is a bi-modal logic L with two unary modalities $\Box$ (always true in the future) and $\blacklozenge$ (possibly true in the past) such that:

$$\blacklozenge\varphi \to \psi \in L \text{ if and only if } \varphi \to \Box\psi \in L$$

As usual, we have $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$ and $\Diamond\varphi := \neg\Box\neg\varphi$

Alternatively, a tense logic is a normal bi-modal logic containing the axioms:

- $p \to \Box\blacklozenge p$
- $p \to \blacksquare\Diamond p$

Kripke-frame for tense logics: $\mathfrak{F} = (X, R, R^{-1})$

Let K_t be the minimal tense logic, $\mathsf{K4}_t = \mathsf{K}_t \oplus \Diamond p \to \Diamond\Diamond p$ and $\mathsf{S4}_t = \mathsf{K4}_t \oplus p \to \Diamond p$

Tense Logics

Tense logics are ‘innocent expansion’ of modal logics

Tense Logics

Tense logics are 'innocent expansion' of modal logics

Lattices of tense logics are substantially different from those of modal logics

Tense Logics

Tense logics are 'innocent expansion' of modal logics

Lattices of tense logics are substantially different from those of modal logics

The lattice NExt(K) contains only 2 co-atoms [Makinson, 1971]
However, there are $2^{\aleph_0}$ co-atoms in NExt(K_t), even in NExt($K4_t$) (see [Chen and Ma, 2024])

Tense Logics

Tense logics are ‘innocent expansion’ of modal logics

Lattices of tense logics are substantially different from those of modal logics

The lattice NExt(K) contains only 2 co-atoms [Makinson, 1971]
However, there are $2^{\aleph_0}$ co-atoms in NExt(K_t), even in NExt($K4_t$) (see [Chen and Ma, 2024])

There are countably many splittings in NExt(K) [Blok, 1978], while
there is exactly one splitting in NExt(K_t) [Kracht, 1992]

Tense Logics

Tense logics are 'innocent expansion' of modal logics

Lattices of tense logics are substantially different from those of modal logics

The lattice NExt(K) contains only 2 co-atoms [Makinson, 1971]
However, there are $2^{\aleph_0}$ co-atoms in NExt(K_t), even in NExt($K4_t$) (see [Chen and Ma, 2024])

There are countably many splittings in NExt(K) [Blok, 1978], while
there is exactly one splitting in NExt(K_t) [Kracht, 1992]

Degree of Kripke-incompleteness in lattices of tense logics?

Degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{K}_t)$

One result about the lattices $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{K}_t)$ from our previous work [Chen, 2025]

Theorem (Blok's Theorem for $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$)

Let $L \in \mathsf{NExt}(\mathsf{K}_t)$ (or $L \in \mathsf{NExt}(\mathsf{K4}_t)$). Then the following are equivalent:

- *L is a union-splitting*
- $\deg(L) = 1$
- $\deg(L) \neq 2^{\aleph_0}$

In this work...

We turn to study the degree of Kripke-incompleteness of $\mathsf{NExt}(\mathsf{S4}_t)$ and show the following:

Theorem (Blok's Theorem for $\mathsf{NExt}(\mathsf{S4}_t)$)

Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. Then the following are equivalent:

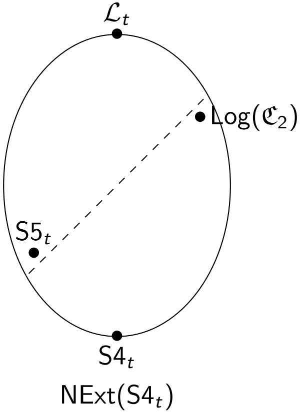
- *L is an iterated splitting in $\mathsf{NExt}(\mathsf{S4}_t)$*
- $\deg(L) = 1$
- $\deg(L) \neq 2^{\aleph_0}$

One more thing: there exists $L \in \mathsf{NExt}(\mathsf{S4}_t)$ which is not a union-splitting but with degree 1

Outline

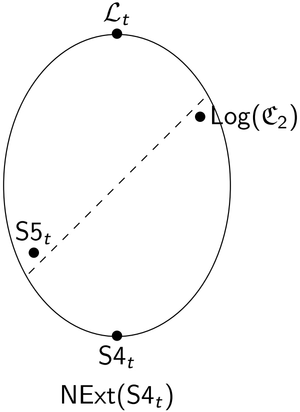
Iterated splittings in NExt($S4_t$)

Recall from [Kracht, 1992] that $\langle \mathrm{Log}(\mathfrak{C}_2), S5_t \rangle$ and $\langle \mathrm{Log}(\circ), \mathcal{L}_t \rangle$ are the only two splitting pairs in NExt($S4_t$), where $\mathfrak{C}_2 = (2, \leq)$ and $S5_t = S4_t \oplus (p \to \Box\Diamond p)$



Iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$

Recall from [Kracht, 1992] that $\langle \mathsf{Log}(\mathfrak{C}_2), \mathsf{S5}_t \rangle$ and $\langle \mathsf{Log}(\circ), \mathcal{L}_t \rangle$ are the only two splitting pairs in $\mathsf{NExt}(\mathsf{S4}_t)$, where $\mathfrak{C}_2 = (2, \leq)$ and $\mathsf{S5}_t = \mathsf{S4}_t \oplus (p \to \Box\Diamond p)$



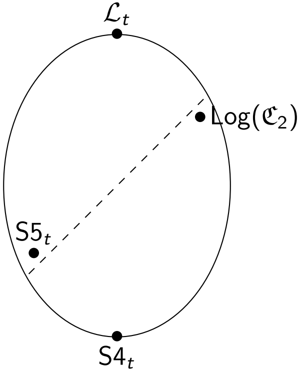
Lemma

Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. Then the following are equivalent:

- *L is an iterated splitting in $\mathsf{NExt}(\mathsf{S4}_t)$*
- $L \in \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$

Iterated splittings in NExt($S4_t$)

Recall from [Kracht, 1992] that $\langle \mathrm{Log}(\mathfrak{C}_2), S5_t \rangle$ and $\langle \mathrm{Log}(\circ), \mathcal{L}_t \rangle$ are the only two splitting pairs in NExt($S4_t$), where $\mathfrak{C}_2 = (2, \leq)$ and $S5_t = S4_t \oplus (p \to \Box\Diamond p)$



Lemma

Let $L \in$ NExt($S4_t$). *Then the following are equivalent:*

- L *is an iterated splitting in* NExt($S4_t$)
- $L \in \mathrm{NExt}(S5_t) \cup \{S4_t\}$

Theorem

Every iterated splitting in NExt($S4_t$) *is strictly Kripke-complete*

Non-iterated splittings in NExt($S4_t$)

Take any non-iterated splitting L in NExt($S4_t$).
It suffices now to prove that $\deg(L) = 2^{\aleph_0}$

To make the idea precise, let us focus on a concrete logic $L_0 = S4_t \oplus \Box\Diamond p \to \Diamond\Box p$

Non-iterated splittings in NExt($S4_t$)

Take any non-iterated splitting L in NExt($S4_t$).
It suffices now to prove that $\deg(L) = 2^{\aleph_0}$

To make the idea precise, let us focus on a concrete logic $L_0 = S4_t \oplus \Box\Diamond p \to \Diamond\Box p$

How to show that L_0 is not strictly complete?

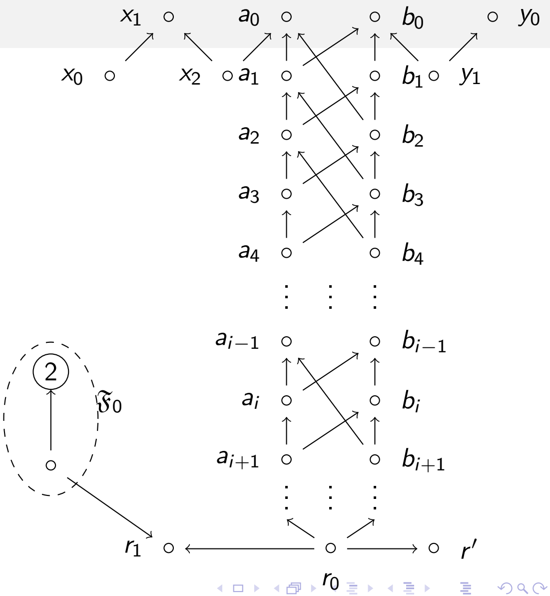
Non-iterated splittings in NExt($S4_t$)

Take any non-iterated splitting L in NExt($S4_t$). It suffices now to prove that $\deg(L) = 2^{\aleph_0}$

To make the idea precise, let us focus on a concrete logic $L_0 = S4_t \oplus \Box\Diamond p \to \Diamond\Box p$

How to show that L_0 is not strictly complete?

Consider the general frame $\mathbb{F} = (\mathfrak{F}, A)$ on the right hand side, where A is the set of all finite and co-finite subsets of X



Non-iterated splittings in NExt($S4_t$)

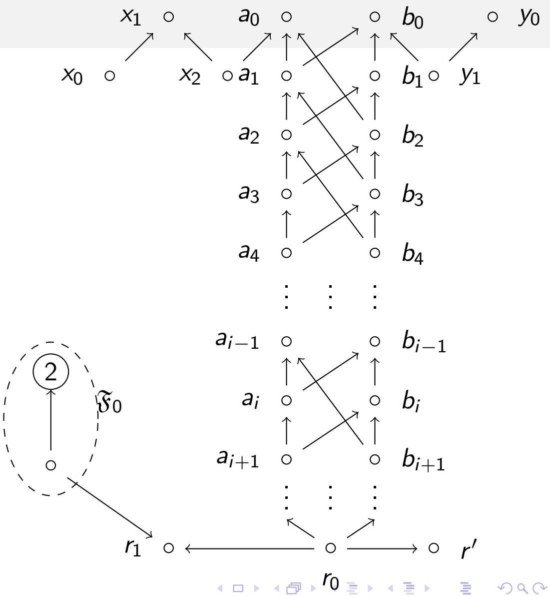
Take any non-iterated splitting L in NExt($S4_t$). It suffices now to prove that $\deg(L) = 2^{\aleph_0}$

To make the idea precise, let us focus on a concrete logic $L_0 = S4_t \oplus \Box\Diamond p \to \Diamond\Box p$

How to show that L_0 is not strictly complete?

Consider the general frame $\mathbb{F} = (\mathfrak{F}, A)$ on the right hand side, where A is the set of all finite and co-finite subsets of X

Let $L' = L_0 \cap \mathrm{Log}(\mathbb{F})$. Then $\Box\Diamond p \to \Diamond\Box p \notin L'$

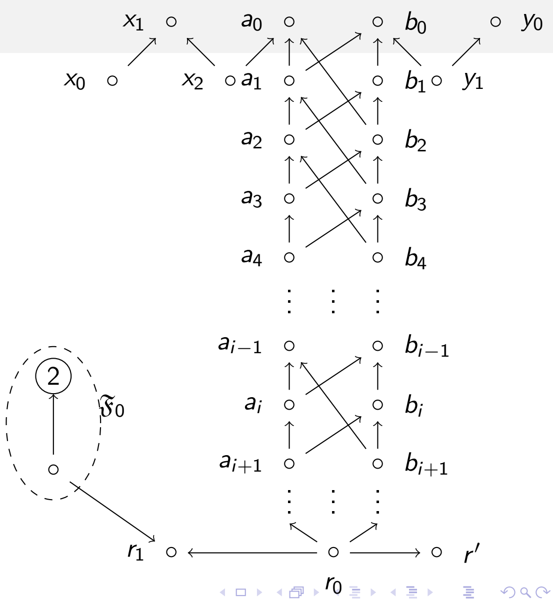


Non-iterated splittings in NExt($S4_t$)

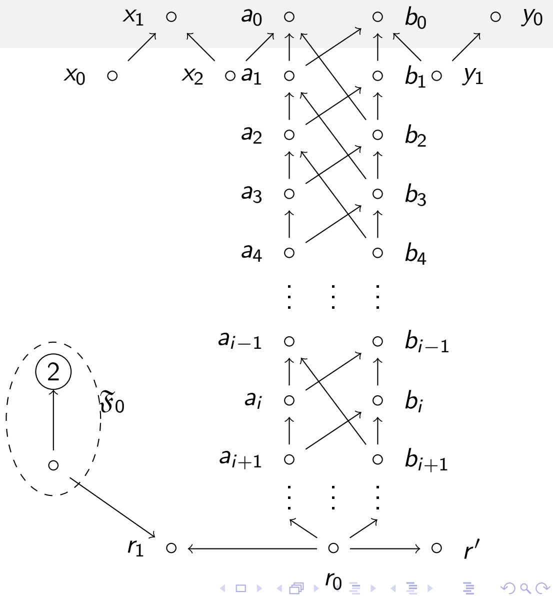
The general $\mathbb{F}$ has several special properties:

Lemma

$\mathrm{Fin}_r(\mathrm{Log}(\mathbb{F})) = \mathrm{Fr}_r(\mathrm{Log}(\mathbb{F})) = \{\circ, \mathfrak{C}_2\}$.



Non-iterated splittings in NExt($S4_t$)



The general $\mathbb{F}$ has several special properties:

Lemma

$\mathrm{Fin}_r(\mathrm{Log}(\mathbb{F})) = \mathrm{Fr}_r(\mathrm{Log}(\mathbb{F})) = \{\circ, \mathfrak{C}_2\}$.

Lemma

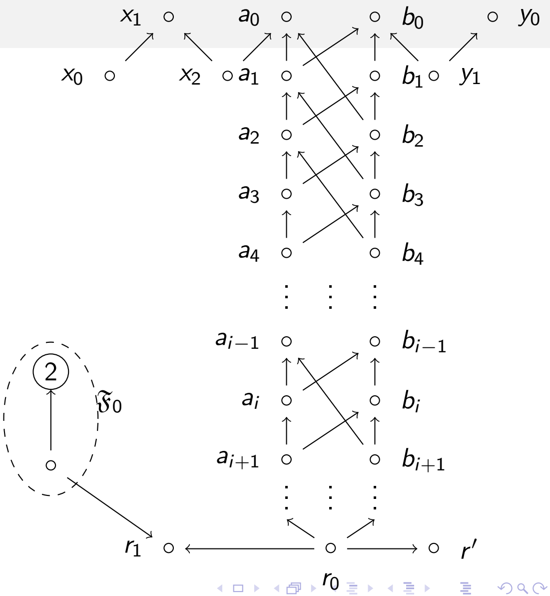
$\mathrm{Fr}(L) = \mathrm{Fr}(L')$.

Theorem

$\deg(L_0) > 1$.

Non-iterated splittings in NExt($S4_t$)

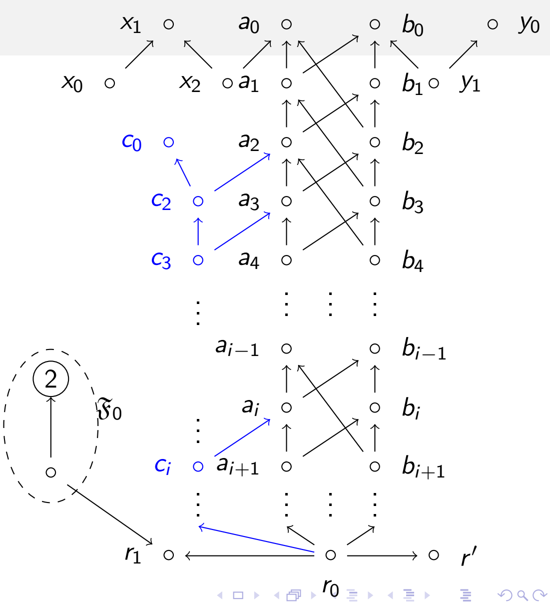
How to construct $2^{\aleph_0}$ logics in $[L_0]_{\mathrm{Fr}}$?



Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L_0]_{\mathrm{Fr}}$?

For each $I \subseteq \omega$, construct $\mathbb{F}_I$ by adding points $\{c_i : i \in I\}$ and the corresponding arrows to $\mathbb{F}$:



Lemma

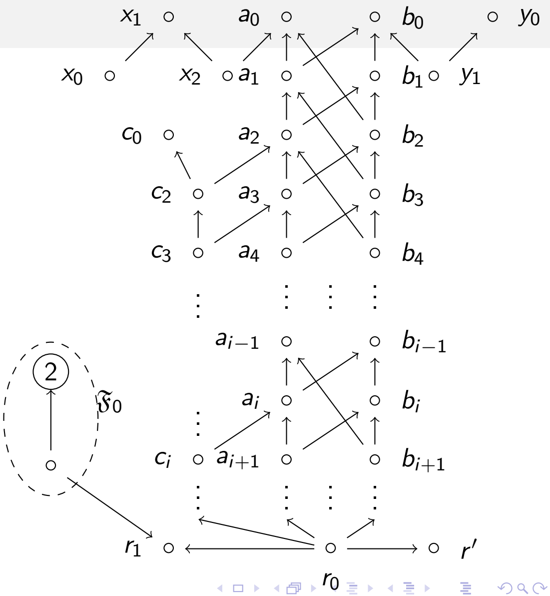
For all distinct $I, J \subseteq \mathbb{Z}^+$,

$$L_0 \cap \mathrm{Log}(\mathbb{F}_I) \neq L_0 \cap \mathrm{Log}(\mathbb{F}_J)$$

As a corollary, we see $\deg(L_0) = 2^{\aleph_0}$

Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L]_{Fr}$ for arbitrarily chosen non-iterated splitting L?



Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L]_{\mathrm{Fr}}$ for arbitrarily chosen non-iterated splitting L?

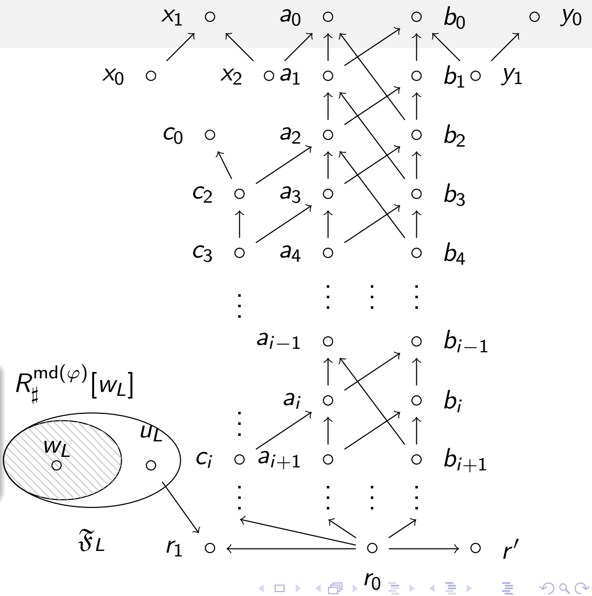
For each L, construct $\mathbb{F}^L_I$ by replacing $\mathfrak{F}_0$ with $\mathfrak{F}_L$, where $\mathfrak{F}_L \in \mathrm{Fin}_r(S4_t)$, $\varphi \in L \setminus S4_t$, $\mathfrak{F}_L, w_L \not\models \varphi$ and $u_L \notin R_\sharp^{\mathrm{md}(\varphi)}[w_L]$

Theorem

For all distinct $I, J \subseteq \mathbb{Z}^+$,

$$L \cap \mathrm{Log}(\mathbb{F}_I) \neq L \cap \mathrm{Log}(\mathbb{F}_J)$$

As a corollary, we see $\deg(L) = 2^{\aleph_0}$ for all non-iterated splittings



Conclusions

Blok's dichotomy theorem is generalized from NExt(K) (also NExt(K_t)) to NExt($S4_t$)

Conclusions

Blok's dichotomy theorem is generalized from NExt(K) (also NExt(K_t)) to NExt($S4_t$)

Future works:

- Degree of Kripke-incompleteness for other lattices of logics, say, NExt($S4.3_t$) or Ext(IPC)
- Degree of incompleteness w.r.t other semantics, for example, topological semantics
- Back to the basic modal case :)

Thanks!

↑ Scan the QR-code for the preprint *Degree of Kripke-incompleteness of Tense Logics*

Bezhanishvili, G., Bezhanishvili, N., and Moraschini, T. (2025).
Degrees of the finite model property: The antidichotomy theorem.
Journal of Mathematical Logic.
DOI: 10.1142/S0219061325500060.

Blok, W. J. (1978).
On the degree of incompleteness in modal logic and the covering relation in the lattice of modal logics.
Technical Report 78-07.
University of Amsterdam.

Chagrova, L. (1998).
On the degree of neighborhood incompleteness of normal modal logics.
In *Advances in Modal Logic*, volume 1 of *CSLI Lecture Notes*, pages 63–72. CSLI Publications.

Chen, Q. (2025).
Degree of kripke-incompleteness in tense logics.

In *Proceedings of The Logic Algebra and Truth Degrees 2025*, Siena, Italy.

Chen, Q. and Ma, M. (2024).
Tabularity and post-completeness in tense logic.
The Review of Symbolic Logic, 17(2):475–492.

Chernev, A. (2022).
Degrees of fmp in extensions of bi-intuitionistic logic.
Master's thesis, University of Amsterdam, Amsterdam.

Dziobiak, W. (1978).
A note on incompleteness of modal logics with respect to neighbourhood semantics.
Bulletin of the Section of Logic, 7(4):185–189.

Fine, K. (1974).
An incomplete logic containing s4.
Theoria, 40(1):23–29.

Kracht, M. (1992).

Even more about the lattice of tense logics.
Archive for Mathematical Logic, 31(4):243–257.

Litak, T. (2004).
Modal incompleteness revisited.
Studia Logica, 76(3):329–342.

Makinson, D. (1971).
Some embedding theorems for modal logic.
Notre Dame Journal of Formal Logic, 12(2):252–254.

Thomason, S. K. (1972).
Semantic analysis of tense logics.
Journal of Symbolic Logic, 37(1):150–158.

Van Benthem, J. F. A. K. (1978).
Two simple incomplete modal logics.
Theoria, 44(1):25–37.