Craig Interpolation for Logics of Negative Modality via Cut-Free Sequent Calculus

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Our Contribution

- proposes cut-free sequent calculi for three expansions of positive intuitionistic propositional logic by negative modalities.
- investigates the Craig interpolation properties of these three logics.

Craig Interpolation

For all formulas φ and ψ, if $\varphi \to \psi$ is a theorem of L then there is a formula χ s.t. both $\varphi \to \chi$ and $\chi \to \psi$ are theorems of L and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

- holds in both CPC and IPC.
- derives: Beth definability theorem (1953) and Robinson joint consistency (1956).
- can be proved via proof theory (Maehara's method), model theory, and algebra.

- Proof theory
- Semantics

Negative Modality

Došen (1986 & 1999)'s investigation:

how can a negation weaker than minimal negation be added to positive IPC?

Adding new binary relation C on a Kripke model for positive IPC.

$$M, w \models {\sim}\varphi \quad \text{iff} \quad \text{for any } v \in W : wCv \text{ implies } M, v \not\models \varphi.$$

negative modality

Syntax of This Talk

$$\mathsf{Form} \ni \varphi ::= p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi, \quad (p \in \mathsf{Prop}).$$

- The absurdity constant $\bot$ and the tautology constant $\top$ are not included.

Kripke Model

$M = (W, \leq, C, V)$ where

- W is a non-empty set of states,
- $\leq$ is a partial-order,
- C is a binary relation on W s.t.

$$\leq \circ C \subseteq C \circ \leq^{-1},$$

For a logic to be closed under a uniform substitution

- $V : \text{Prop} \to \mathcal{P}(W)$ s.t.

$$w \in V(p) \text{ and } w \leq v \text{ imply } v \in V(p).$$

Kripke Semantics

$$
\begin{array}{lll}
M, w \models p & \text{iff } w \in V(p), \\
M, w \models \varphi \wedge \psi & \text{iff } M, w \models \varphi \text{ and } M, w \models \psi, \\
M, w \models \varphi \vee \psi & \text{iff } M, w \models \varphi \text{ or } M, w \models \psi, \\
M, w \models \varphi \to \psi & \text{iff for any } v \in W \\
& \quad w \leq v \text{ and } M, v \models \varphi \text{ imply } M, v \models \psi,
\end{array}
$$

$$M, w \models {\sim}\varphi \quad \text{iff} \quad \text{for any } v \in W : wCv \text{ implies } M, v \not\models \varphi.$$

- φ is valid in a Kripke model $M = (W, \leq, C, V)$
  $\Leftrightarrow$ for all $w \in W$: $M, w \models \varphi$.

Logics N and ND

- N: the set of valid formulas in all Kripke models,
- ND: the set of valid formulas in all serial Kripke models, i.e., Kripke models $M = (W, \leq, C, V)$ s.t.

for any $w \in W$, there is $v \in W$ satisfying wCv.

- The absurdity constant $\perp$ is definable in ND by $\sim(p \to p)$ but is not definable in N.

C may be $\emptyset$.

Star Model

Kripke model with C being a function $*$.

$M = (W, \leq, *, V)$ where

- W, $\leq$, and V are as before,
- $*$ is a function from W to W s.t.

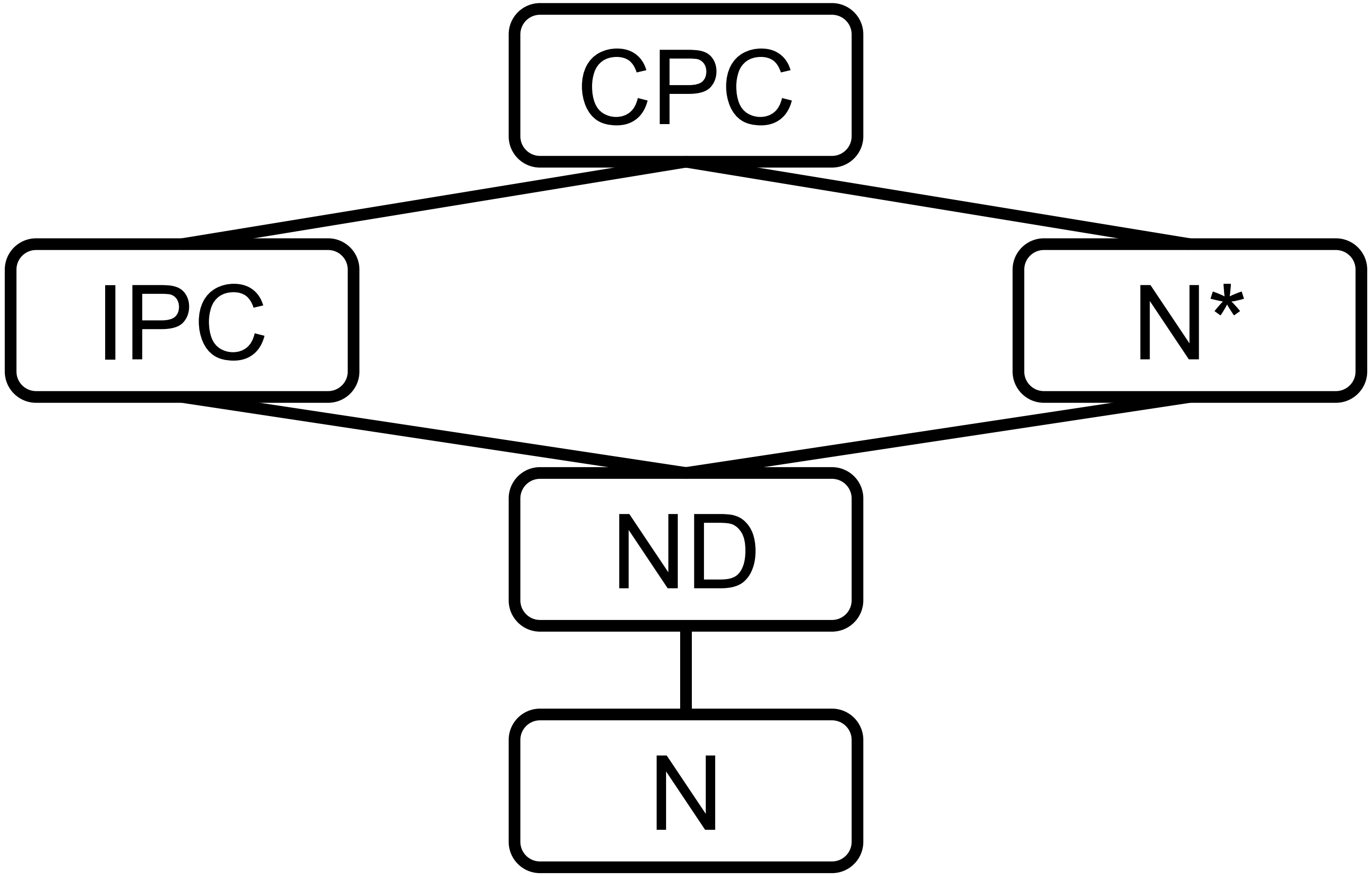
$$w \leq v \text{ implies } v^* \leq w^*.$$

For a logic to be closed under a uniform substitution

$$M, w \models \sim\varphi \quad \text{iff } M, w^* \not\models \varphi.$$

Logic N*

- N*: the set of valid formulas in all star models.



Model Classes

- $\mathbb{M}_{\mathbf{N}}$: the class of all Kripke models,
- $\mathbb{M}_{\mathbf{ND}}$: the class of all serial Kripke models,
- $\mathbb{M}_{\mathbf{N}^*}$: the class of all star models.

Proof Theory

- Hilbert systems H(N) & H(ND) (cf. Došen 1986).
- Hilbert system H(N*) (cf. Drobyshevich & Odintsov 2013).
- Sequent calculus for a logic with negative modality (Lahav, Marcos, & Zohar 2017).

✓Two negative modalities (``all” &``some”),

✓No implication.

- H(N): Positive part of H(IPC) +

$$(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi),$$

From $\varphi \to \psi$, we may infer $\sim\psi \to \sim\varphi$.

- G(N): Positive part of Maehara (1954)'s mLJp +

$$\frac{\varphi \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\varphi},$$

where $\sim\Delta = \{\sim\chi \mid \chi \in \Delta\}$.

- φ must exist in this rule.

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From $\varphi \rightarrow \psi$, we may infer $\sim\psi \rightarrow \sim\varphi$.

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$$\frac{\varphi \vee \psi \Rightarrow \varphi, \psi}{\sim\varphi, \sim\psi \Rightarrow \sim(\varphi \vee \psi)} .$$

- φ must exist in this rule.

- H(ND): Positive part of H(IPC) +

$$(\sim\varphi \wedge \sim\psi) \rightarrow \sim(\varphi \vee \psi),$$

From $\varphi \rightarrow \psi$, we may infer $\sim\psi \rightarrow \sim\varphi$.

$$\sim(\varphi \rightarrow \varphi) \rightarrow \psi.$$

- G(ND): Positive part of Maehara (1954)'s mLJp +

$$\frac{\Phi \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Phi},$$

where Φ is either a singleton or ∅.

- H(ND): Positive part of H(IPC) +

$$(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi),$$

From $\varphi \to \psi$, we may infer $\sim\psi \to \sim\varphi$.

$$\sim(\varphi \to \varphi) \to \psi.$$

- G(ND): Positive part of Maehara (1954)'s mLJp +

$$\frac{\square \Rightarrow \varphi \to \varphi}{\sim(\varphi \to \varphi) \Rightarrow \square}.$$

- H(N*): Positive part of H(IPC) +

$(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi)$,

From $\varphi \to \psi$, we may infer $\sim\psi \to \sim\varphi$.

$\sim(\varphi \to \varphi) \to \psi$.

$\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi)$,

$\sim((\varphi \to \varphi) \to \psi)$.

- G(N*): Positive part of Maehara (1954)'s mLJp +

$$\frac{\Phi \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Phi}.$$

No restriction.

- H(N*): Positive part of H(IPC) +

$$(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi),$$

From $\varphi \to \psi$, we may infer $\sim\psi \to \sim\varphi$.

$$\sim(\varphi \to \varphi) \to \psi.$$

$$\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi),$$

$$\sim((\varphi \to \varphi) \to \psi).$$

- G(N*): Positive part of Maehara (1954)'s mLJp +

$$\frac{\varphi, \psi \Rightarrow \varphi \wedge \psi}{\sim(\varphi \wedge \psi) \Rightarrow \sim\varphi, \sim\psi}.$$

Fact (Došen 1986, Drobyshevich & Odintsov 2013)

Let $\Lambda \in \{\mathbf{N}, \mathbf{ND}, \mathbf{N}^*\}$.

$$\mathbb{M}_\Lambda \models \varphi \quad \text{iff} \quad \mathsf{H}(\Lambda) \vdash \varphi.$$

Euipollentness of Two Systems (New!)

Cut is necessary for ⇒.

Let $\Lambda \in \{\mathbf{N}, \mathbf{ND}, \mathbf{N}^*\}$.

$$\mathsf{H}(\Lambda) \vdash \varphi \quad \text{iff} \quad \mathsf{G}(\Lambda) \vdash \Rightarrow \varphi.$$

Cut Elimination (New!)

By ``extended cut rule'' (Kashima 2009).

Let $\Lambda \in \{\mathbf{N}, \mathbf{ND}, \mathbf{N}^*\}$. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\Lambda)$, then there is a derivation in $\mathsf{G}(\Lambda)$ whose root is $\Gamma \Rightarrow \Delta$ with no application of (Cut).

Two Points

1. Treatment to intuitionistic multi-succedent sequent calculus.
2. Reformulation of Craig interpolation.

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For all formulas φ and ψ, if $\varphi \to \psi$ is a theorem of L then there is a formula χ s.t. both $\varphi \to \chi$ and $\chi \to \psi$ are theorems of L and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

- Usually, the cut elimination establishes the Craig interpolation property via Maehara (1961)'s method.

If $\Gamma \Rightarrow \Delta$ is derivable in LK, then for any partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of $\Gamma \Rightarrow \Delta$, there is a formula χ s.t.

- both $\Gamma_1 \Rightarrow \Delta_1, \chi$ and $\chi, \Gamma_2 \Rightarrow \Delta_2$ are derivable in LK and
- $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$.

- Maehara method is not applied straightforwardly to intuitionistic multi-succedent sequent calculus.
- Two Solutions:

1. Restricting the form of a partition (normal partition) (cf. Kowalski & Ono 2017)

✓ Bi-intuitionistic logic (Kowalski & Ono 2017)

2. Extending the notion of an interpolant (Mints' interpolant) (cf. Mints 2001)

✓ Bi-intuitionistic tense logic (BiSKt) (Ono & S. 2022)

- Recall that our calculi are based on mLJp.

Two Points

1. Treatment to intuitionistic multi-succedent sequent calculus.
2. Reformulation of Craig interpolation.

If $G \vdash\Rightarrow \varphi \to \psi$, then there is a formula χ s.t. both $G \vdash\Rightarrow \varphi \to \chi$ and $G \vdash\Rightarrow \chi \to \psi$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

- If neither $\top$ nor $\bot$ exists in the syntax, this claim does not hold even in CPC.

Consider the case where $\mathrm{Prop}(\varphi) \cap \mathrm{Prop}(\psi) = \emptyset$.

If $\mathsf{G} \vdash \Rightarrow \varphi \to \psi$, then there is a formula χ s.t. both $\mathsf{G} \vdash \Rightarrow \varphi \to \chi$ and $\mathsf{G} \vdash \Rightarrow \chi \to \psi$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

- If neither $\top$ nor $\bot$ exists in the syntax, this claim does not hold even in CPC.
- Craig interpolation is sensitive to the syntax.
- Recall that our syntax does not contain $\top$ or $\bot$.

Reformulation

If G $\vdash \Rightarrow \varphi \to \psi$, then one of the following holds:

- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) \neq \emptyset$, then there is a formula χ s.t. both G $\vdash \Rightarrow \varphi \to \chi$ and G $\vdash \Rightarrow \chi \to \psi$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$,
- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) = \emptyset$, then either G $\vdash \varphi \Rightarrow$ or G $\vdash \Rightarrow \psi$.

- If $\top$ and $\bot$ exist, the two formulations are equivalent.
- Seki's method: Craig interpolation for CPC, IPC, and substructural logics with this formulation.

Craig Interpolation for ND and N* (New!)

Let $\Lambda \in \{\mathbf{ND}, \mathbf{N}^*\}$. If $\mathsf{G}(\Lambda) \vdash \Rightarrow \varphi \to \psi$, then one of the following holds:

- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) \neq \emptyset$, then there is a formula χ s.t. both $\mathsf{G}(\Lambda) \vdash \Rightarrow \varphi \to \chi$ and $\mathsf{G}(\Lambda) \vdash \Rightarrow \chi \to \psi$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$,
- if $\mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi) = \emptyset$, then either $\mathsf{G}(\Lambda) \vdash \varphi \Rightarrow$ or $\mathsf{G}(\Lambda) \vdash \Rightarrow \psi$.

- For ND: normal partition with Seki’s method.
- For N*: Mints’ interpolant with Seki’s method.

$\perp$ is not definable in N.

Failure of Craig interpolation for N (New!)

All the following items hold:

- $\mathsf{G}(\mathbf{N}) \vdash\Rightarrow \sim(q \to q) \to \sim p$,
- $\mathsf{Prop}(\sim(q \to q)) \cap \mathsf{Prop}(\sim p) = \emptyset$,
- $\mathsf{G}(\mathbf{N}) \nvdash \sim(q \to q) \Rightarrow$ and $\mathsf{G}(\mathbf{N}) \nvdash\Rightarrow \sim p$.

- Cut elimination ensures that it is impossible to derive a sequent of the form $\Gamma \Rightarrow$ in general.

- In the following, we expand the syntax by $\bot$.
- Let $G(\Lambda_\bot)$ be the calculus obtained by adding to $G(\Lambda)$ the following rule:

$$\frac{}{\bot \Rightarrow}$$

Craig Interpolation for the expansions by ⊥ (New!)

Let $\Lambda \in \{\mathbf{N}_\perp, \mathbf{ND}_\perp, \mathbf{N}^*_\perp\}$. If $\mathsf{G}(\Lambda) \vdash \varphi \to \psi$, then there is a formula χ s.t. both $\mathsf{G}(\Lambda) \vdash \varphi \to \chi$ and $\mathsf{G}(\Lambda) \vdash \chi \to \psi$ and $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

- For $N_\perp$ and $ND_\perp$: employing normal partition.
- For $N^*_\perp$: employing Mints' interpolant.

No need of Seki's method.

G(N), G(ND), G(N*)

Our Contribution

- proposes cut-free sequent calculi for three expansions of positive intuitionistic propositional logic by negative modalities.
- investigates the Craig interpolation properties of these three logics.

- N does not satisfy the property,
- ND and N* satisfy the property,
- $N_\bot$, $ND_\bot$, and $N^*_\bot$ satisfy the property.

Thank You!