The Rule Dichotomy Property via Stable Canonical Rules

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Outline

1. Introduction
2. Stable canonical rules
3. Main results
4. Conclusion

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Admissible rules

Our aim is to study admissible rules in modal logics. By rules, we mean multi-conclusion rules (or, multi-conclusion consequence relations).

Definition

Let L be a modal logic. A rule Γ/Δ is admissible in L or L-admissible if for any substitution σ,

$$\forall \gamma \in \Gamma \ (\sigma\gamma \in L) \text{ implies } \exists \delta \in \Delta \ (\sigma\delta \in L).$$

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A historical overview:

- [Friedman, 1975] asked whether the admissibility of a given inference rule in the intuitionistic logic IPC is decidable.
- [Rybakov, 1990, 1992] showed that this is the case for IPC and a large class of transitive modal and superintuitionistic logics.
- [Ghilardi, 1999, 2000] gave an alternative proof by connecting the admissibility to projective formulas and unification.
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Decidability of Admissibility

Admissiblity is shown to be decidable in many transitive modal logics, including K4, S4, and GL. However, it is mostly unknown in the non-transitive case.

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The rule dichotomy property

Definition

Let L be a modal logic and $\mathcal{R}$ be a class of rules.

1. $\mathcal{R}$ has the rule dichotomy property over L if every rule in $\mathcal{R}$ is either L-admissilbe or L-equivalent to an assumption-free rule.
2. L has the rule dichotomy property if every rule is L-equivalent to a set of rules that are either L-admissible or assumption-free.

If $\mathcal{R}$ is complete over L (namely, any rule is axiomatizable over L by rules in $\mathcal{R}$) and $\mathcal{R}$ has the rdp over L, then L has the rdp.

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The rule dichotomy property

If L has the fmp and the rdp in a computable way, then the admissibility is decidable in L.

- Let ρ be a given rule.
- Compute a set of rules that are either admissible or assumption-free.
- For each assumption-free one among them, check if it is admissible.

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Stable canonical rules and formulas

As the class $\mathcal{R}$ of rules, while [Jeřábek, 2009] used canonical rules, we will use stable canonical rules, introduced in [Bezhanishvili et al., 2016].

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Stable canonical rules

Each pair of a finite modal algebra $\mathfrak{A}$ and a subset $D \subseteq A$ induces a stable canonical rule $\rho(\mathfrak{A}, D)$, which has the following semantic characterization.

Theorem (Bezhanishvili et al., 2016)

For any modal algebra $\mathfrak{B}$,

$$\mathfrak{B} \not\models \rho(\mathfrak{A}, D) \text{ iff } \mathfrak{A} \hookrightarrow_D \mathfrak{B},$$

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The rule dichotomy in wK4

Theorem

Stable canonical rules have the rule dichotomy property over wK4.

The proof involves complicated combinatorics, adapting the proof for K4 in [Bezhanishvili et al., 2016]. The main idea is to use the semantic characterization of stable canonical rules and the duality, which allows us to work with modal spaces, where it is easier to apply combinatorics.

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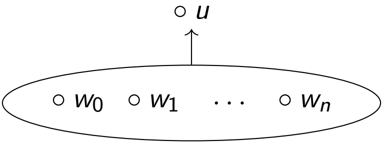
The rule dichotomy in K

As for K, however, we showed that the stable canonical rules $\rho(\mathcal{F}_n, \emptyset)$ are neither K-admissible nor K-equivalent to an assumption-free rule, where $\mathcal{F}_n$ is as depicted below.



The rule dichotomy in K

To show the inadmissibility, we constructed new theorems from $\rho(\mathcal{F}_n, \emptyset)$ and applied a consequence of the characterization of admissible rules in [Metcalfe, 2012]: if a rule derives new theorems, then it is inadmissible.

To show that they are not equivalent to assumption-free rules, we used the following characterization.

Proposition (Jeřábek, 2009)

For any rule ρ, the following are equivalent.

1. *Validity of ρ is preserved by generated subframes (i.e., upsets),*
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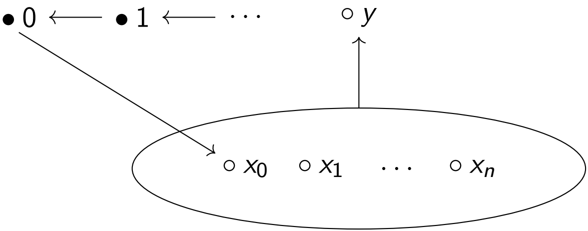
Let $\mathcal{G}_n$ be the following modal space where the clopen sets are finite without y or cofinite with y.



Then, $\mathcal{F}_n$ is an upset of $\mathcal{G}_n$. However, there is no stable map from $\mathcal{G}_n$ to $\mathcal{F}_n$, namely, $\mathcal{G}_n \models \rho(\mathcal{F}_n, \emptyset)$, whereas $\mathcal{F}_n \not\models \rho(\mathcal{F}_n, \emptyset)$.

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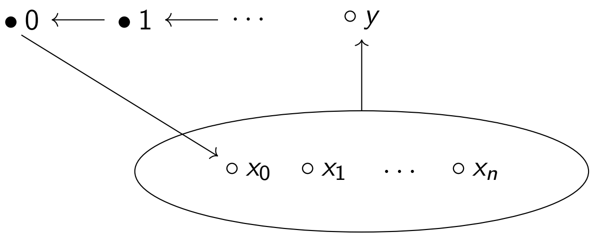

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The rule dichotomy in K

Theorem

Stable canonical rules do not have the rule dichotomy property over K.

Admissible stable rules

Moreover, we obtained some sufficient conditions for a stable canonical rule to be admissible or inadmissible in K. Though they are complicated and ad hoc, they yield a characterization of admissible stable rules, namely, stable canonical rules with $D = \emptyset$.

Corollary

A stable rule $\rho(\mathcal{F}, \emptyset)$ is K*-admissible iff there is no $x \in \mathcal{F}$ such that xRy for all $y \in \mathcal{F}$.*

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Results summary

The case $L = \mathsf{K4}$.

$\mathcal{R} =$	{canonical rules}	{stable canonical rules}
$\mathcal{R}$ is complete over L	✓	✓
$\mathcal{R}$ has the rdp over L	✓	✓
L has the rdp	✓	✓
admissibility is decidable in L	✓	✓

Results summary

The case $L = \mathrm{wK4}$.

$\mathcal{R} =$	{canonical rules}	{stable canonical rules}
$\mathcal{R}$ is complete over L	?	?
$\mathcal{R}$ has the rdp over L	?	✓
L has the rdp	?	?
admissibility is decidable in L	?	?

Results summary

The case $L = \mathsf{K}$.

$\mathcal{R} =$	{canonical rules}	{stable canonical rules}
$\mathcal{R}$ is complete over L	$\times$	$\checkmark$
$\mathcal{R}$ has the rdp over L	$\checkmark$	$\times$
L has the rdp	?	?
admissibility is decidable in L	?	?

Remark and future work

- Given the successful generalization of the rdp of stable canonical rules from K4 to wK4, we also tried for pre-transitive logics. However, there seems to be an essential combinatorial difficulty there.
- Related to the unification problem, restricting to rules with the conclusion $\perp$ will not help for this method. This is because when calculating the corresponding set of stable canonical rules, we take filtration with $\mathrm{Sub}(\Gamma \cup \Delta)$.
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