Observation Algebras

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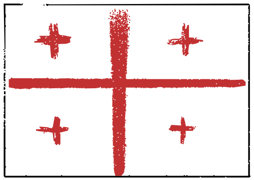
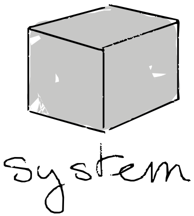
Tskaltubo



Part I:

The model

An illustration



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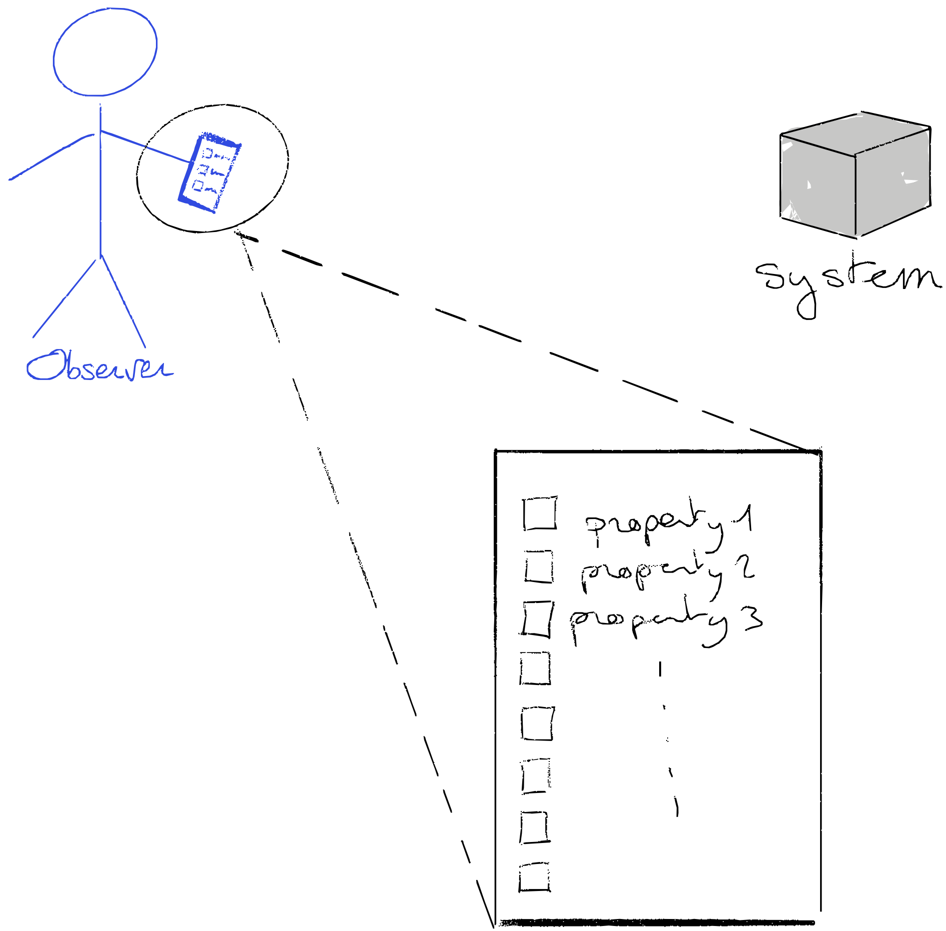

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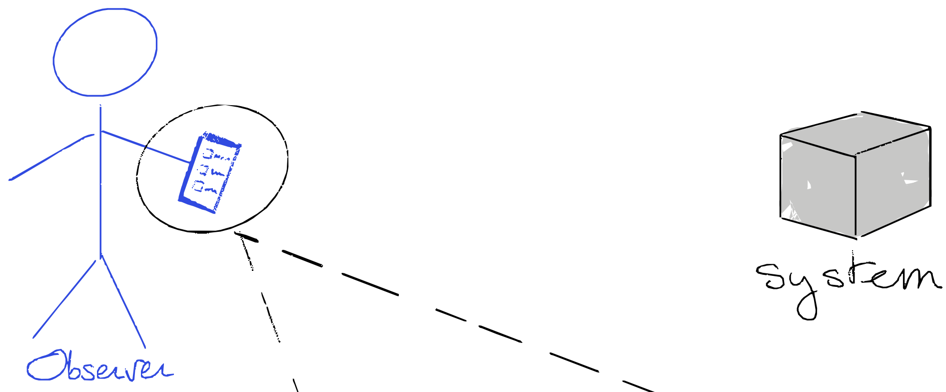
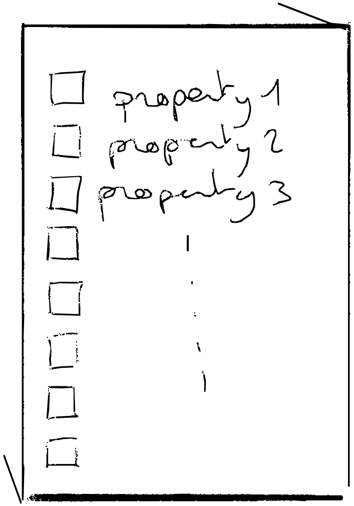
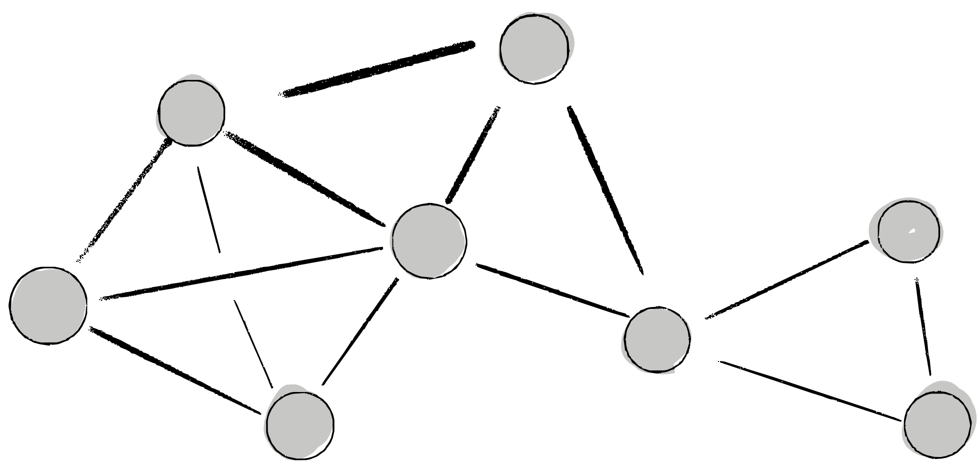


- Can only observe positive properties
- may omit some valid properties (partial view)

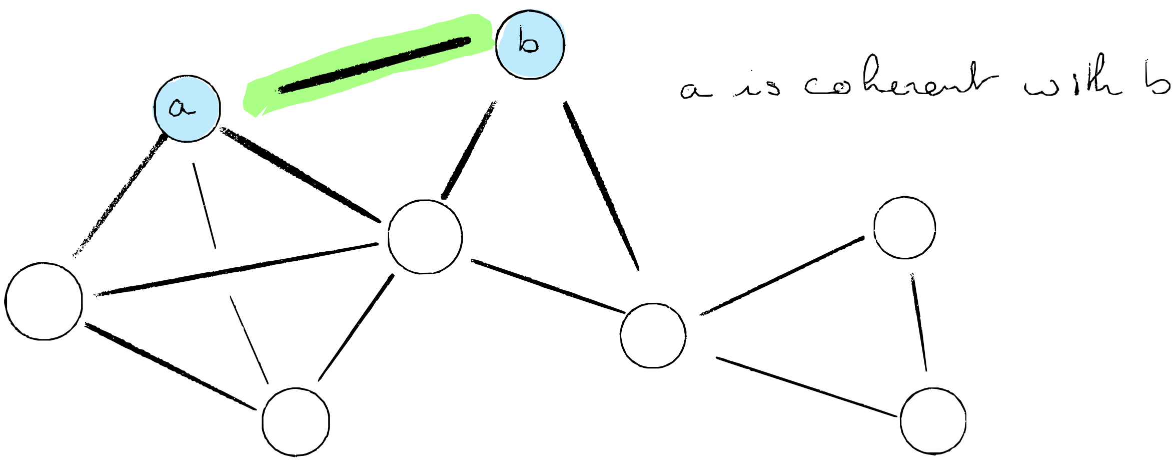


Consider some undirected graph



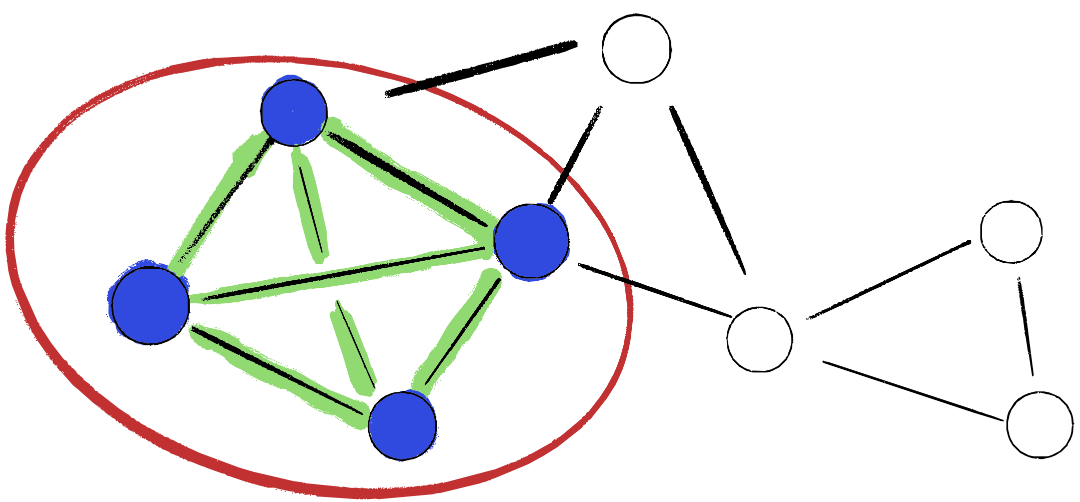
Consider some undirected graph

- its edges are interpreted as a coherence relation, and its vertices as chromic observations



Consider some undirected graph

- its edges are interpreted as a coherence relation, and its vertices as atomic observations
- an observation is a coherent set of atomic observations (ie a clique)



Part II :

The language

Observation Logic

$$\varphi, \psi ::= \top \mid \bot \mid a \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi$$

$\in V$

$\models \subseteq Obs \times Formulae(V)$

$$x \models \top \qquad \frac{a \in x}{x \models a} \qquad \frac{\exists i : x \models \varphi_i}{x \models \varphi_1 \vee \varphi_2} \qquad \frac{\forall i \; x \models \varphi_i}{x \models \varphi_1 \wedge \varphi_2}$$

$$\boxed{\frac{\forall y,\ x \cup y \in Obs \;\&\; x \cup y \models \varphi \Rightarrow x \cup y \models \psi}{x \models \varphi \to \psi}}$$

Definitions

- $[\![\varphi]\!] = \{x \in Obs \mid x \models \varphi\}$
- $\varphi \simeq \psi$ iff $[\![\varphi]\!] = [\![\psi]\!]$

Observation Logic

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Definitions

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- $\varphi \simeq \psi$ iff $[\![\varphi]\!] = [\![\psi]\!]$

Notice the following

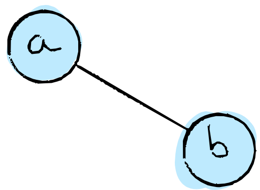
1) $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \qquad [\![\bot]\!] = \emptyset$

2) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\top]\!] = Obs$

3) $\forall X \subseteq Obs, \quad X \subseteq [\![\varphi \to \psi]\!] \Leftrightarrow X \cap [\![\varphi]\!] \subseteq [\![\psi]\!]$

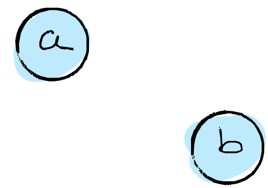
Let's axiomatize!

- We want to define axiomatically a relation $\equiv$ on formulae such that $\varphi \equiv \psi \iff \varphi \approx \psi$
- This relation will depend on the graph:



$[\![a \wedge b]\!] = \{\{a, b\}\}$

$a \wedge b \not\approx \bot$



$[\![a \wedge b]\!] = \emptyset = [\![\bot]\!]$

$a \wedge b \approx \bot$

- Although our sets of axioms will be infinite, for every pair $\varphi \equiv \psi$ there is a finite proof tree witnessing the identity.

Let's axiomatize!

① Obvious axioms : Heyting Algebra

$$\varphi \vee \bot \equiv \varphi \qquad \varphi \vee \psi \equiv \psi \vee \varphi \qquad \varphi \vee (\psi \vee \xi) \equiv (\varphi \vee \psi) \vee \xi$$

$$\varphi \wedge \top \equiv \varphi \qquad \varphi \wedge \psi \equiv \psi \wedge \varphi \qquad \varphi \wedge (\psi \wedge \xi) \equiv (\varphi \wedge \psi) \wedge \xi$$

COMMUTATIVE MONOIDS

$$\varphi \vee (\varphi \wedge \psi) \equiv \varphi \qquad \varphi \wedge (\varphi \vee \psi) \equiv \varphi$$

BOUNDED LATTICE

$$\varphi \to \varphi \equiv \top$$

$$\varphi \wedge (\varphi \to \psi) \equiv \varphi \wedge \psi$$

$$\psi \wedge (\varphi \to \psi) \equiv \psi$$

$$\varphi \to (\psi \wedge \xi) \equiv (\varphi \to \psi) \wedge (\varphi \to \xi)$$

Let's axiomatize!

1. Obvious axioms = Heyting Algebra
2. No less obvious = encode the edges

$\forall a, b \in V,$ $\boxed{(a,b) \notin E \Rightarrow a \wedge b \equiv \bot}$

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1. Obvious axioms = Heyting Algebra
2. No less obvious = encode the edges

$$\forall a, b \in V, \quad \boxed{(a, b) \notin E \implies a \wedge b \equiv \bot} \; (E)$$

__Lemma__ for formulae φ, ψ without $\to$, if $\equiv$ is defined as BOUNDED LATTICE + (E), then

$$\varphi \equiv \psi \iff \varphi \approx \psi$$

Let's axiomatize!

1. Obvious axioms = Heyting Algebra
2. No less obvious = encode the edges
3. General properties of cliques

$$\forall x \in Obs, \quad \left(\bigwedge x\right) \to (\varphi \vee \psi) \equiv \left(\bigwedge x \to \varphi\right) \vee \left(\bigwedge x \to \psi\right)$$

(looks like $A \to (B \vee C) \equiv (A \to B) \vee (A \to C)$.)

$$\forall x \in Obs, \ \forall a \notin x, \quad \bigwedge x \to a \equiv \left(\bigwedge x \to \bot\right) \vee a$$

(looks like $A \to B \equiv \neg A \vee B$)

Let's axiomatize!

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$\forall x \in Obs, \forall a \notin x, \quad \bigwedge x \to a \equiv (\bigwedge x \to \bot) \vee a$

$\bigwedge x$ exists when x is a finite set of nodes

$\bigwedge \{a_1, \dots, a_n\} = a_1 \wedge \dots \wedge a_n \qquad \bigwedge \emptyset = \top$

Let's axiomatize!

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$$\forall x \in Obs_f \quad (\bigwedge x) \to (\varphi \vee \psi) \equiv (\bigwedge x \to \varphi) \vee (\bigwedge x \to \psi)$$

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$\bigwedge x$ exists when x is a finite set of nodes

$$\bigwedge \{a_1, \dots, a_n\} = a_1 \wedge \dots \wedge a_n \qquad \bigwedge \emptyset = \top$$

$$Obs_f = \left\{ x \subseteq V \;\middle|\; \begin{array}{l} x \text{ finite} \\ \forall a \neq b \in x, \ (a,b) \in E \end{array} \right\}$$

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With the axioms listed so far, we do have

$$\varphi \equiv \psi \implies \varphi \approx \psi$$

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With the axioms listed so far, we do have

$$\varphi \equiv \psi \implies \varphi \approx \psi$$

... but the converse does not hold!

As no further "obvious" axioms have been found, let us restrict our graphs.

Part III:

Classes of graphs.

The FAN property

Finite AntiNeighbourhood.

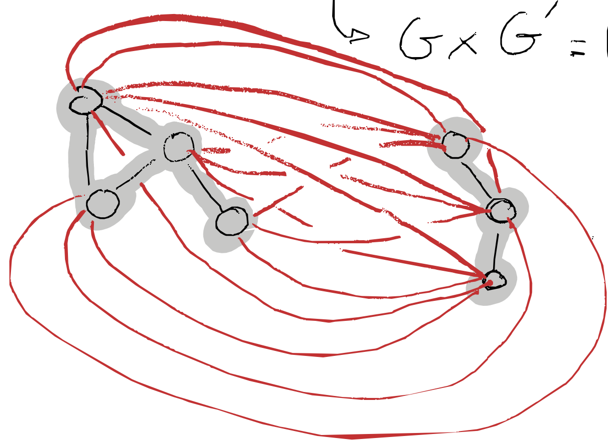
G is FAN iff $\forall a \in V$ the set $\{ b \in V \mid (a,b) \notin E \}$ is finite

every node is disconnected from finitely many other nodes.

Example: . finite graphs

. ∞ products of finite graphs

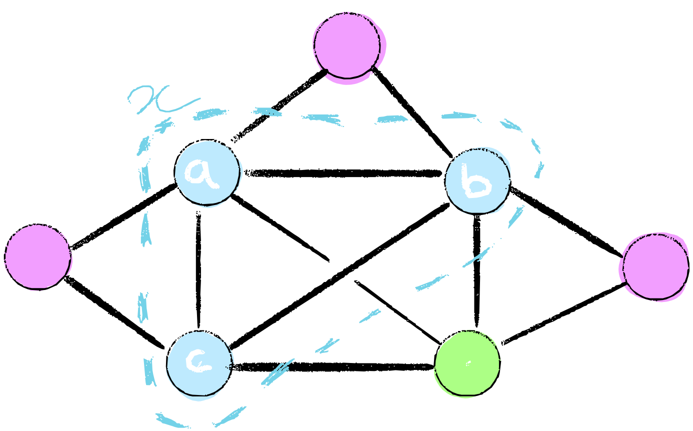
$G \times G' = (V \uplus V', E \cup E' \cup V \times V'$
$\cup V' \times V)$



FAN, axiomatically

$\forall x \in \mathrm{Obs}_f$

$$\bigwedge x \to \bot \;\equiv\; \bigvee \{a \in V \mid \exists b \in x : (a,b) \notin E\}$$



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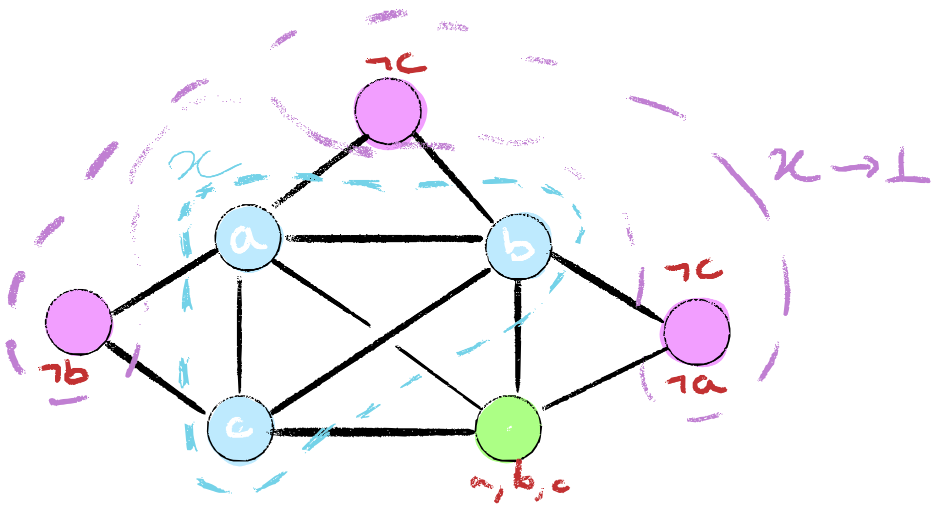

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$\forall x \in \mathrm{Obs}_f$

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<u>Theorem</u> If G is FAN, then $\forall \varphi, \psi$

$$\varphi \equiv \psi \iff \varphi \approx \psi$$

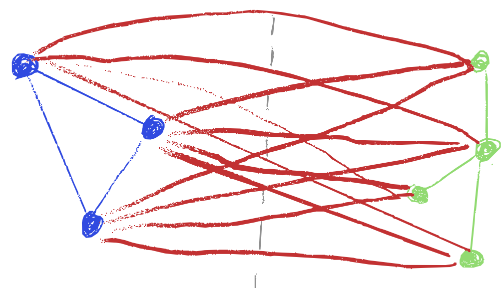
Other classes of graphs

$\hookrightarrow$ ∞ anticliques ($\simeq \omega$)

cliques are either $\emptyset$ or singleton sets

$[\![\varphi]\!]$ denotes either a finite or a cofinite set of vertices.

$\hookrightarrow$ (∞) products of "tractable" graphs $(G_i)_{i \in I}$



compile formulas into DNF/CNF of "term vectors"

i.e. maps from $i \in I$ to formulas over G_i

Example A memory model

Two (∞) sets of variables $\overset{\text{boolean}}{V_{\mathbb{B}}}$ $\overset{\text{numeric}}{V_{\mathbb{N}}}$

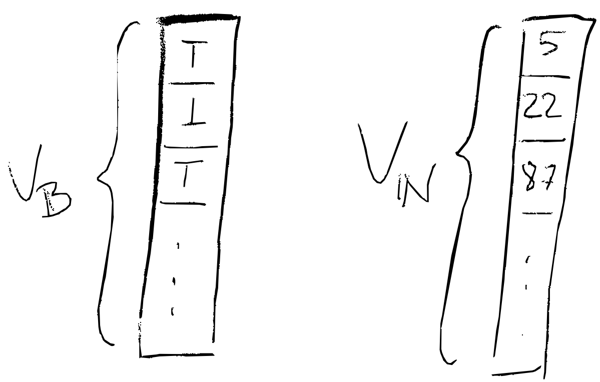
Atomic observation $x_{\mathbb{B}} = \bot$ $x_{\mathbb{B}} = \top$ $x_{\mathbb{N}} = n$

Expressible predicates $x_{\mathbb{B}} \neq \bot := x_{\mathbb{B}} = \top$

More subtle: $x_{\mathbb{N}} \neq n$ $\left(\equiv \underbrace{\bigvee_{m \neq n} x_{\mathbb{N}} = m}_{\infty \text{ formula}}\right)$

But $x = 5 \to x \neq 2$

ie $(x=5) \to ((x=2) \to \bot) \equiv \top$ is derivable



Conclusion

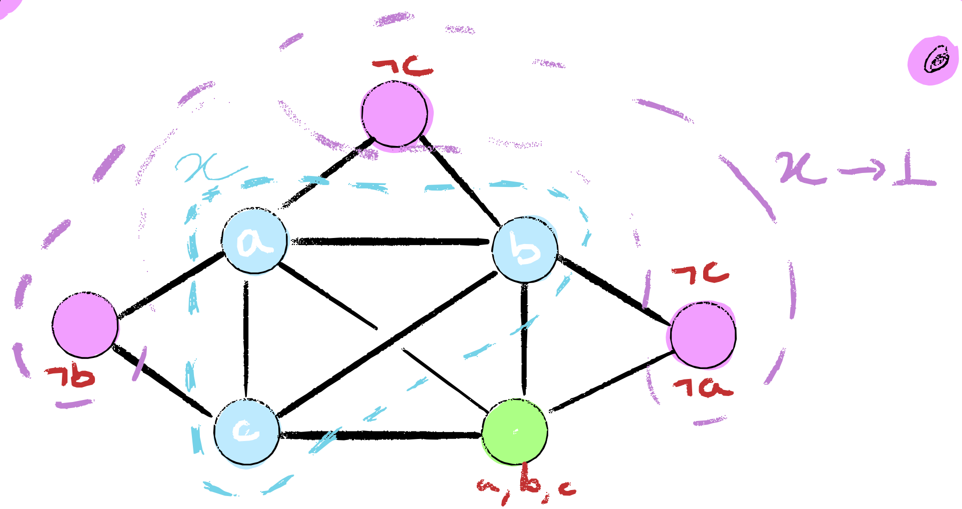
Conclusion & future endeavours

Recap:
- semantic model as sets of cliques
- a series of axiomatisations of $\varphi \simeq \psi$ for certain classes of graphs.
- can be used to model e.g. memory

fully formalised in Rocq

Future:
- more axiomatisations for more graphs
- Horn theory, FO ...
- applications to program verification
  ↳ combined with a model of dynamic behaviour, observation algebra can provide syntax + sem for the static part of the model

That's all Folks !



site web : paul.brunet-zamansky.fr

Interpretation

Imagine a set of events Evt, with a satisfaction relation $\models \subseteq Evt \times V$, checking whether an event satisfies an atomic observation (vertex/predicate).

An observation (clique) is satisfied by an event if all its vertices are satisfied:

$$\boxed{e \models x \iff \forall v \in x,\ e \models v}$$

Let us order observations

$G = (V, E)$ $\quad$ $Obs = \{ x \subseteq V \mid \forall a \neq b \in x,\ (a,b) \in E \}$

$x, y \in Obs$

$$\boxed{x \precsim y \underset{\text{definition}}{\iff} x \supseteq y}$$

$x \precsim y$ means y is "more general" than x

Indeed, if $x \precsim y$ and $e \models x$, then $e \models y$.

Operations on observations

① Join $\boxed{a \leq x \ \& \ b \leq x \iff a \vee b \leq x}$

$$x \vee y := x \cap y$$

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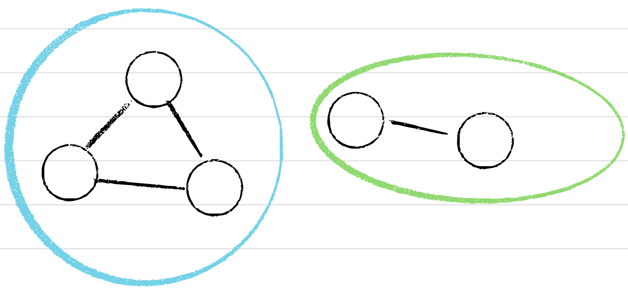
② Meet $\boxed{x \leq a \ \& \ x \leq b \iff x \leq a \wedge b}$

... does not always exist



($x \wedge y$ does not exist whenever $\exists a \in x \ \exists b \in y : (a,b) \notin E$)

If it exists, $x \wedge y := x \cup y$

ie if $x \cup y \in Obs$

Note: there is a largest clique ($\emptyset$) but no least clique

Beyond FAN: $\mathbb{N}$ (aka ω)

$\overset{0}{\bullet} \quad \overset{1}{\bullet} \quad \overset{2}{\bullet} \quad \cdots \qquad \overset{n}{\bullet} \quad \cdots \qquad (\mathbb{N}, \emptyset)$ no edges

very not FAN

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no edges

very not FAN

$\neg$ for every vertex $n \in \mathbb{N}$

$$[\![n \vee (n \to \bot)]\!] = \{\{m\} \mid m \in \mathbb{N}\}$$

$$\hookrightarrow \quad n \vee (n \to \bot) \equiv m \vee (m \to \bot)$$

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② for more complicated reasons:

$\forall X \subseteq_f V,$

$$((\bigvee X) \to \bot) \to \bot \equiv \bigvee X$$

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$\forall X \subseteq_f V,$

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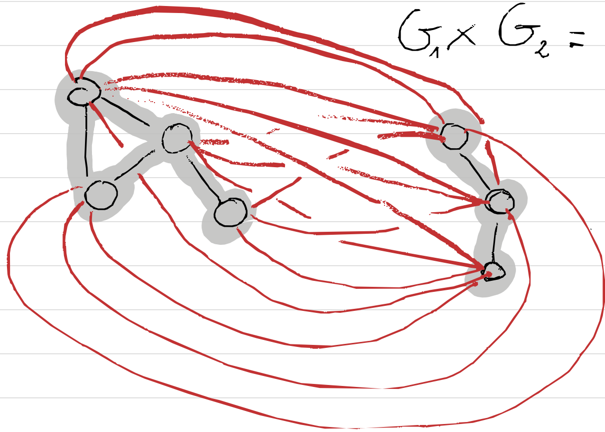
Almost like a boolean algebra except
① $n \vee (n \to \bot) \not\equiv \top$
② $\varphi \to \bot \to \bot \not\equiv \varphi$ for arbitrary formulae

$\underline{\text{Thm}}_{\text{I}}$ $\quad \varphi \equiv \psi \iff \varphi \approx \psi$

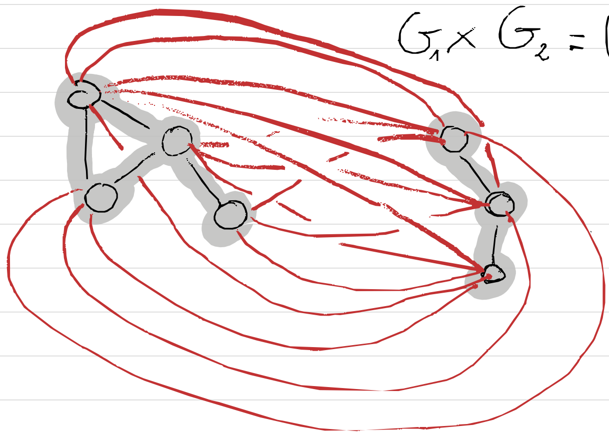
Products



$$G_1 \times G_2 = (V_1 \uplus V_2 \, , E_1 \cup E_2 \cup V_1 \times V_2 \cup V_1 \times V_2)$$

In general, for an arbitrary family $(G_i)_{i \in I}$ we can define $\prod_{i \in I} G_i$.

Products



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In general, for an arbitrary family $(G_i)_{i \in I}$ we can define $\prod_{i \in I} G_i$.

① $\forall \varphi, \psi \in \mathrm{Formulae}(V_i)$, $\varphi \equiv_i \psi \Rightarrow \varphi = \psi$

Some axiomatisation of G_i

Products

for almost all i
$v_i = \perp$

(2) Term vector : finitely supported function v
of (dependant) type $\bigwedge i : I .\ \mathrm{Formulae}(V_i)$

$\sqcap v = \bigwedge_{i \in |v|} v_i \qquad \sqcup v = \bigvee_{i \in |v|} v_i$

$$(u \to v)_i = \begin{cases} u_i \to v_i & i \in |v| \cap |u| \\ v_i & i \in |v| \setminus |u| \\ u_i \to \perp & i \in |u| \setminus |v| \end{cases}$$

Products

for almost all i
$v_i = \bot$

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$$\boxed{(\sqcap u) \to (\sqcup v) \equiv \sqcup (u \to v)}$$

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for almost all i
$V_i = \bot$

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$$\boxed{(\sqcap u) \to (\sqcup v) \equiv \sqcup (u \to v)}$$

<u>Thm</u> If $\forall i \quad \varphi \equiv_i \psi \Leftrightarrow \varphi \approx_i \psi$, then

$\varphi \equiv \psi \quad \Leftrightarrow \quad \varphi \approx \psi$