Axiomatization and Decidability of Tense Information Logic

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Overview

1. Introduction
2. Motivation
3. Results
4. Strong completeness proof
5. Decidability of TIL

Tense Information logic

Language of MIL: $\varphi \ := \ \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathsf{sup} \rangle \varphi\psi$

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$\mathfrak{M}, x \Vdash \langle \mathsf{inf} \rangle \varphi\, \psi$ **iff** $\exists y, z : \ \mathfrak{M}, y \Vdash \varphi, \ \mathfrak{M}, z \Vdash \psi, \ x = \mathsf{inf}\{y, z\}$

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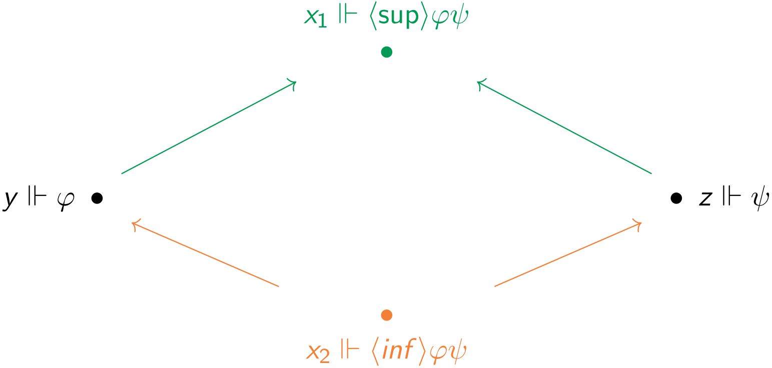
$\mathit{TIL} \ := \ \{ \ \varphi \mid \mathfrak{M} \Vdash \varphi$ for every poset model $\}$



$x_1 \Vdash \langle \mathsf{sup} \rangle \varphi\psi$
$y \Vdash \varphi$
$z \Vdash \psi$
$x_2 \Vdash \langle inf \rangle \varphi\psi$

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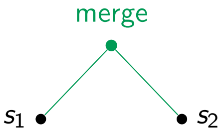

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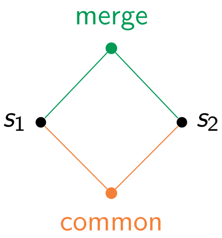

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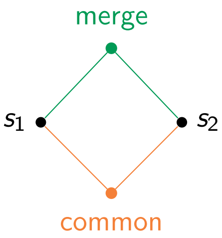

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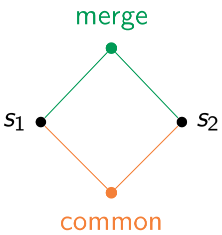

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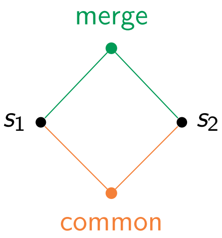

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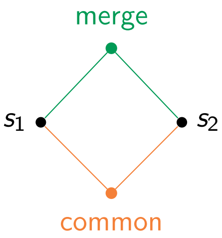

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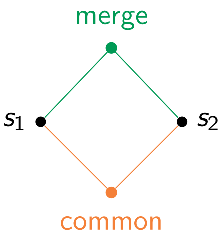

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  → what if we also add ⟨inf⟩?
- TIL on lattices (Wang & Wang [3]) use *nominals*

  → can we obtain completeness *without* them?

Motivation

- Van Benthem [2]:

  Question: "Are there interesting general axioms that link $\langle \mathsf{inf} \rangle \varphi\psi$ to $\langle \mathsf{sup} \rangle \varphi\psi$?"

Results at a glance

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- Completeness on preorder frames
- Decidability via completeness

Axiomatization

Remark

$$\begin{aligned}
\mathsf{P}\,\varphi &:= \langle \mathsf{sup} \rangle \varphi \top && \textit{past looking diamond} \\
\mathsf{H}\,\varphi &:= \neg \langle \mathsf{sup} \rangle \neg \varphi \top && \textit{past looking box} \\
\mathsf{F}\,\varphi &:= \langle \mathsf{inf} \rangle \varphi \top && \textit{future looking diamond} \\
\mathsf{G}\,\varphi &:= \neg \langle \mathsf{inf} \rangle \neg \varphi \top && \textit{future looking box}
\end{aligned}$$

Axiomatization

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\begin{aligned}
&(\text{Re.}) \quad (p \wedge q \to \langle sup \rangle pq) \\
&(4) \quad (\mathrm{P}\,\mathrm{P}\,p \to \mathrm{P}\,p) \\
&(\text{Co.}) \quad (\langle sup \rangle pq \to \langle sup \rangle qp) \\
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Axiomatization

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$$\text{(Sy.)} \quad (p \to \mathsf{G}\mathsf{P}\,p) \wedge (p \to \mathsf{H}\mathsf{F}\,p)$$

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TIL is the least normal modal logic containing all axioms.

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Theorem (Soundness)

$$TIL \subseteq \mathbf{TIL}$$

Proof sketch: completeness

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- Step-by-step method

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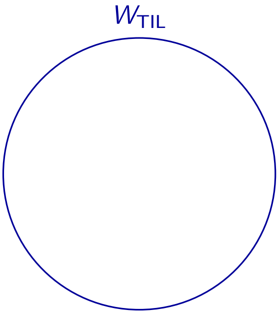

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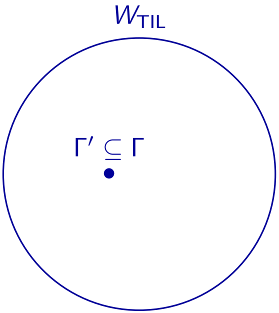

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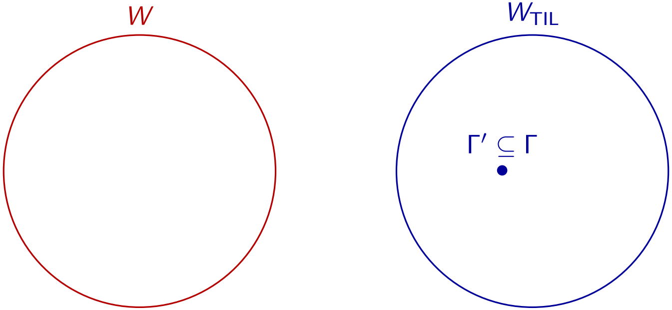

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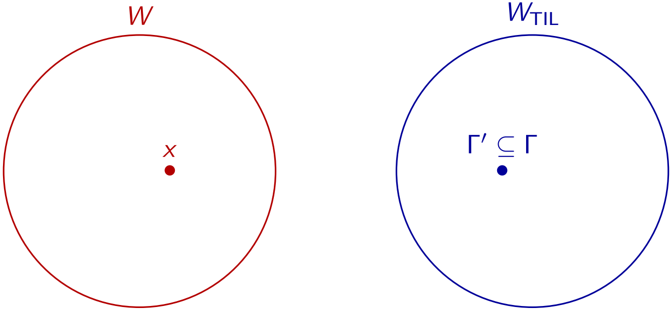

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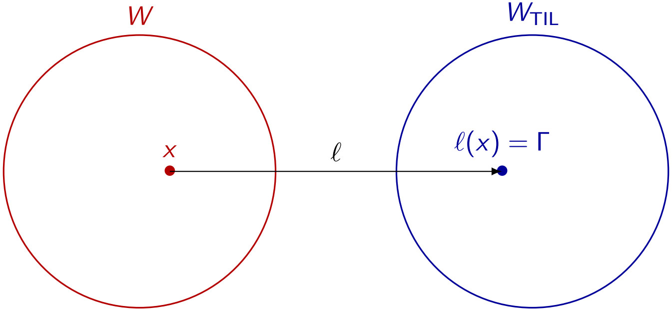

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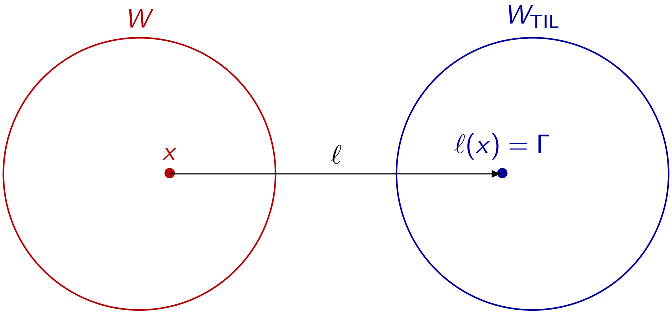

- $\mathfrak{M}, x \Vdash \chi \Leftrightarrow \chi \in l(x)$

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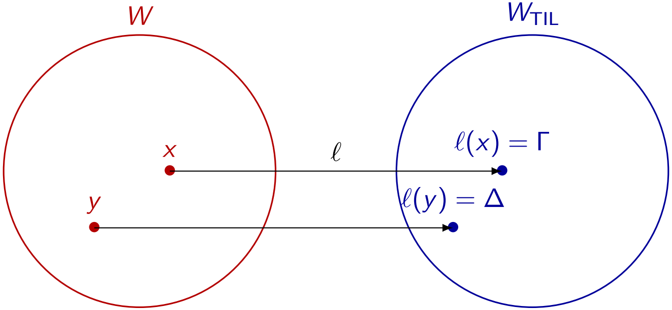

- $\mathfrak{M}, x \Vdash \chi \Leftrightarrow \chi \in l(x)$
- $\ell : W \to W_{\mathsf{TIL}}$

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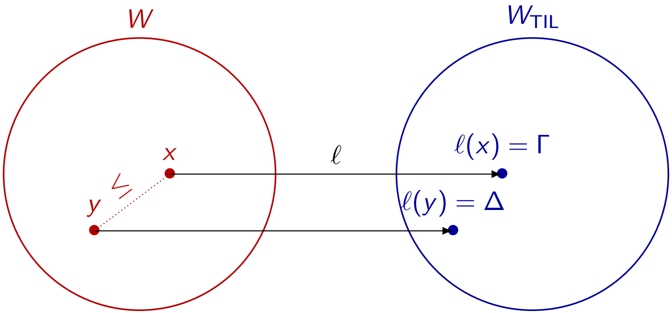

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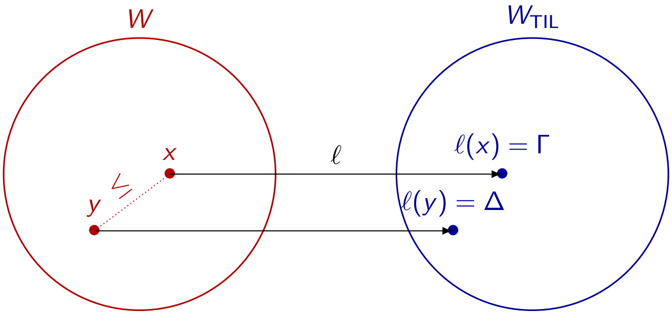

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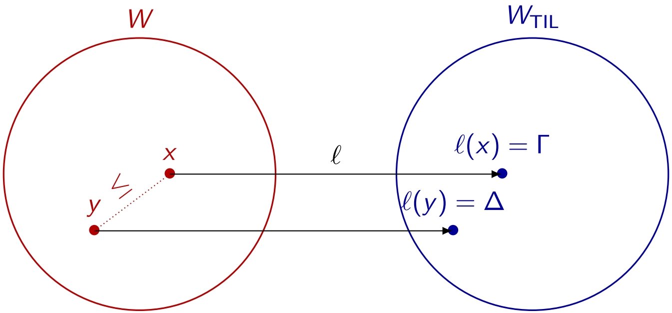

- $\mathfrak{M}, x \Vdash \chi \Leftrightarrow \chi \in I(x)$
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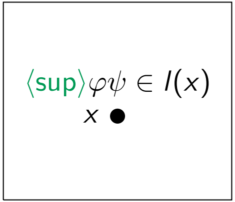
$\langle \text{sup} \rangle$- and $\neg\langle \text{inf} \rangle$-repair



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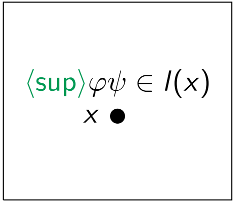

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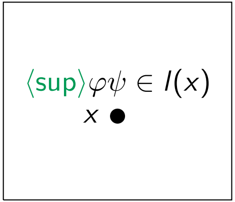

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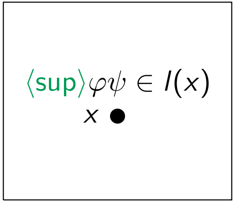

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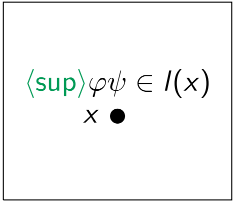

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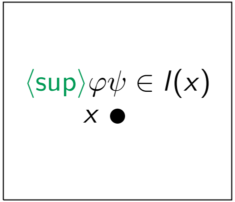

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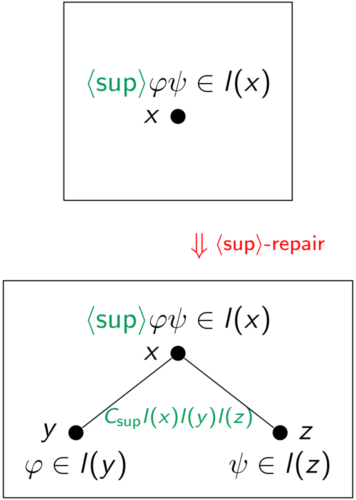

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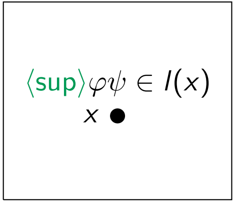
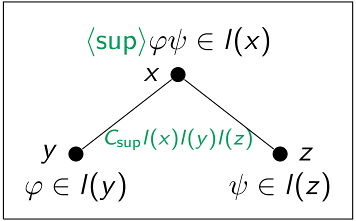
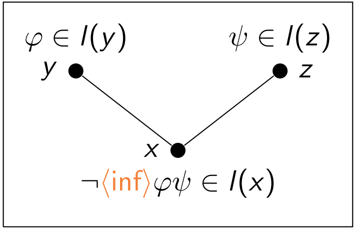


⇓ ⟨sup⟩-repair





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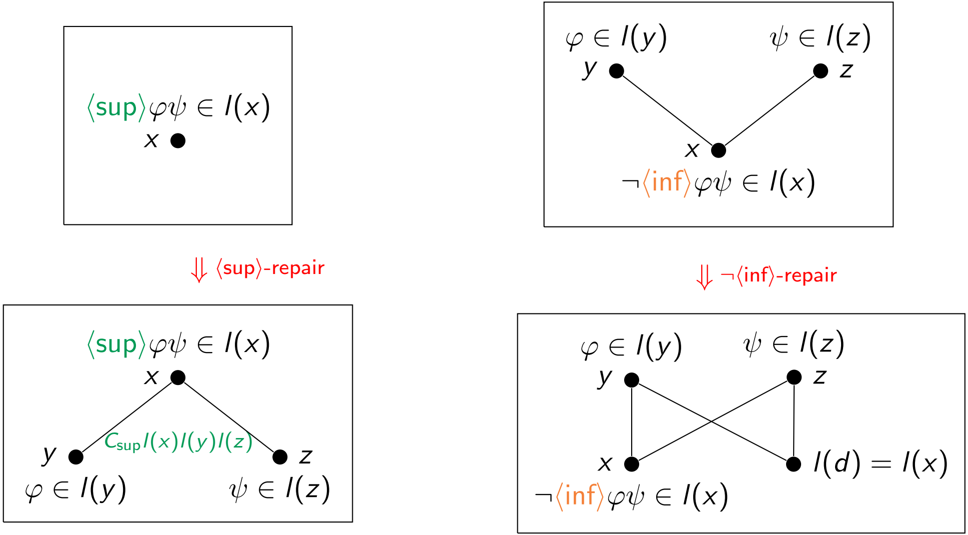

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- Truth lemma for labeled points holds for limit model

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$\mathfrak{M}, x \Vdash \langle \mathsf{inf} \rangle \varphi \psi \quad$ **iff** $\quad \exists y, z : \ \mathfrak{M}, y \Vdash \varphi, \ \mathfrak{M}, z \Vdash \psi, \ x \in \mathsf{inf}\{y, z\}$

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Definition

$TIL_{\mathsf{pre}} \ := \ \{ \ \varphi \mid \mathfrak{M} \Vdash \varphi$ for every preorder model $\}$

Theorem

$TIL_{pre} \ = \ TIL \ = \ \mathbf{TIL}$

Preorders

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Theorem (Completeness + soundness)

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TIL *admits filtration w.r.t.* $\tilde{C}$*-frames*
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Proof.

TIL $=$ **TIL** $= \mathrm{Log}(\tilde{C}_F)$
TIL finitely axiomatisable + admits filtration w.r.t. $\tilde{C}$ □

References

Knudstorp, S.: Modal Information Logics: Axiomatizations and Decidability. Journal of Philosophical Logic **52**(6), 1723–1766 (2023). 10.1007/s10992-023-09724-5

van Benthem, J.: Truth Maker Semantics and Modal Information Logic. Draft manuscript (2017). Available at `https://eprints.illc.uva.nl/id/eprint/1590/`

Wang, X., Wang, Y.: Modal Logics over Lattices. In: Annals of Pure and Applied Logic, **176**(4). Elsevier BV (2025). 10.1016/j.apal.2025.103553

Open questions and future work

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Enables us to translate weak positive logic (WPL) into TIL