A Coalgebraic Semantics for Fischer Servi Logic

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Summary

1. Coalgebras and coalgebraic semantics for (intuitionistic) modal logics;
2. Representations for Fischer-Servi logic;
3. Some consequences and applications of these results.

Coalgebras

Definition (Coalgebra)

Let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$ an endofunctor on $\mathbb{C}$.
A **coalgebra** for F is a pair $(C, \alpha : C \to FC)$

Definition (Coalgebra morphism)

Let (C, α) and (C', β) be coalgebras on the functor F. A **coalgebra morphism** is an arrow $f : C \to C'$ in $\mathbb{C}$ such that $\beta \circ f = Ff \circ \alpha$.

Coalgebras for classical modal logic

- Consider a Kripke frame (W, R), with $R \subseteq W \times W$. Then $R[-] : W \to \mathcal{P}(W)$ is the function mapping a point to its set of successors.
- So any frame can be given as a coalgebra $(W, R : W \to \mathcal{P}W)$ (and vice-versa)
- For $f : W \to W'$, $\mathcal{P}f$ maps a set to its direct image under f. The coalgebra morphisms on $\mathcal{P}$ correspond exactly to p-morphisms.

Another way of saying this: the categories $\mathbf{KFr}$ of Kripke frames and p-morphisms, and $\mathbf{CoAlg}(\mathcal{P})$ of coalgebras of the (covariant) powerset functor, are *equivalent*.

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Descriptive frames

A descriptive general frame is a Kripke frame with additional structure:

Definition

A **descriptive general frame** is a triple (X, R, A) where X is a Stone space, $A = \text{Clop}(X)$, and $R \subseteq X \times X$ such that:

1. $R[x]$ is closed for every $x \in X$
2. If $U \in Clop(X)$, then $R^{-1}[U] \in Clop(X)$

$\mathbf{DG}$ is the category of descriptive general frames with continuous p-morphisms.

Vietoris spaces

Definition

Let X be a Stone space. $V(X)$, consisting of the set of non-empty closed sets of X, is the Vietoris hyperspace of X, given by a topology with subbasis:

$$[U] = \{C \in V(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V(X) : C \cap V \neq \emptyset\},$$

where U, V are clopen subsets of X.

V is an endofunctor on $\mathbf{Stone}$. We have:

Theorem

The categories $\mathbf{DG}$ *and* $\mathbf{CoAlg}(V)$ *are equivalent.*

Coalgebraic semantics

Such a result gives us a *general coalgebraic semantics*. Advantages:

1. Ease in finding notions such as bisimulations;
2. Ease in constructing universal objects like (duals) of free algebras.

Hence it would be desirable to have similar results in other settings. Where can one find them?

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Intuitionistic Modal Logic

We can look at intuitionistic modal logics; look at the language:

$$\mathcal{L} = \wedge \mid \vee \mid \Box \mid \top \mid \bot.$$

The most basic such logic is axiomatized over IPC with $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box\top = \top$.

This can be interpreted over positive Kripke frames: triples $(P, \leq, R)$ where:

1. $(P, \leq)$ is a poset;
2. (P, R) is a Kripke frame;
3. $R = \leq \circ R \circ \leq$.

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Descriptive general frames for $\mathsf{IML}_\Box$

Definition

A $\Box$-**Esakia space** is a triple $(X, \leq, R)$ where $(X, \leq)$ is an Esakia space and $R \subseteq X \times X$ such that:

(i) Whenever U is a clopen upset, then $\Box_R U$ is a clopen upset, where $\Box_R U = \{x \in X : R[x] \subseteq U\}$

(ii) For each $x \in X$, $R[x]$ is a closed upset.

$$R = \leq \circ R \circ \leq$$

Finding the right functor

Definition

Let $V^{\uparrow}(X) := \{C \subseteq X | C$ is a closed upset $\}$ with a topology given by $[U], \langle X - V\rangle$ and the order given by reverse inclusion, resulting in $(V^{\uparrow}(X), \supseteq)$

Then $V^{\uparrow}$ is an endofunctor on **Pries**.

and $\Box$-Esakia spaces are in 1-1 correspondence with Priestley morphisms $R : X \to V^{\uparrow}(X)$ defined by $x \mapsto R[x]$

$V^{\uparrow}$ works for positive modal logic over $\Box$, where we work in **Pries** and the morphisms are continuous monotone maps.

However, for $\mathsf{IML}_{\Box}$ we have the problem that not all Priestley morphisms will be p-morphisms – there are coalgebra morphisms that are not morphisms between $\Box$ Esakia spaces.

The problem is implication

- By Esakia duality, a DL homomorphism $f : D \to D'$ between Heyting algebras is a HA homomorphism iff $f^{-1} : X_{D'} \to X_D$ is a p-morphism.
- Given X an Esakia space and a coalgebra $(X, f : X \to V^{\uparrow}(X))$, we can transform this coalgebra into a coalgebra for another functor that forces the resulting map to be a p-morphism.

We do this through g-openness

Definition

Let $f : X \to Y$ and $g : Y \to Z$ be Priestley morphisms. We say f is g-**open** if f^{-1} preserves relative pseudocomplments of the form $g^{-1}[U] \to g^{-1}[V]$ for U, V clopen. i.e.

$$f^{-1}(g^{-1}[U] \to g^{-1}[V]) = f^{-1}(g^{-1}[U]) \to f^{-1}(g^{-1}[V])$$

We say $S \subseteq X$ is g-open if the inclusion is g-open.

Definition

Let $g : X \to Y$ be a map between Priestley spaces. Then

$$V_g(X) = \{C \subseteq X | C \text{ is closed, rooted, and g-open } \}$$

With the topology given as before.

The functor V_G

Definition 3.5 Let $g : X \to Y$ be a Priestley morphism. The *g-Vietoris complex* $(V_\bullet^g(X), \leq_\bullet)$ over X, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X), ...)$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

(i) $V_0(X) = Y$ and $V_1(X) = X$;

(ii) $r_0 = g$;

(iii) For $i > 1$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;

(iv) For $i > 0$ $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

- V_G then denotes the projective limit of this family.

Coalgebras for $\mathrm{IML}_{\Box}$

The following was the main result from Almeida & Bezhanishvili (2024):

Theorem

The category $\mathbf{CoAlg}(V_G(V^{\uparrow}(-)))$ *is equivalent to the category of* $\Box$*-Esakia spaces with modal p-morphisms.*

There, several generalizations and extensions were proposed. But crucially, the methods there do not apply, without modification to Fischer-Servi logic.

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Fischer-Servi Logic

Definition (Axiomatisation of **IK**)

An algebra $(H, \wedge, \vee, \rightarrow, \Box, \Diamond, \top, \bot)$ is called an $\mathbf{IK}$-algebra if $(H, \wedge, \vee, \rightarrow, \top, \bot)$ is a Heyting algebra, and it satisfies the following modal axioms:

1. $\Box\top = \top$	2. $\Diamond\bot = \bot$
3. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
A. $\Diamond(a \rightarrow b) \leq \Box a \rightarrow \Diamond b$	**B**. $\Diamond a \rightarrow \Box b \leq \Box(a \rightarrow b)$

Definition (Kripke frames)

A K_{FS}-*frame* is a Kripke frame $(X, \leq, R)$ such that

1. $(R\circ \leq) \subseteq (\leq \circ R)$
2. $(\geq \circ R) \subseteq (R\circ \geq)$

Descriptive Frames for Fischer-Servi Logic

Definition (IK-space)

A ***IK**-space* is a modal Esakia space (X, R) such that the following conditions hold:

(T1) $R[x]$ is closed;
(T2) $R[\uparrow x]$ is a closed upset;
(T3) If U is a clopen upset, then $\Diamond_R U$ and $\Box_{(\leq \circ R)} U$ are clopen upsets;
(T4) $R[x] = R[\uparrow x] \cap {\downarrow} R[x]$.

We can also look at R as the intersection of $R_\Box$ and $R_\Diamond$ defined by

$$R_\Diamond[x] = {\downarrow} R[x] \quad \text{and} \quad R_\Box[x] = R[\uparrow x].$$

Finding the right functors

The problem: one of the fundamental axioms involves implications between the added elements!

$$\textbf{B}.\ \Diamond a \to \Box b \leq \Box(a \to b)$$

Our solution: a step-by-step approach

We can start by dealing with axioms 1-4, which do not involve implications. This logic corresponds to the following frames:

Definition

A $\Box\Diamond$-*frame* is a triple $(X, R_\Box, R_\Diamond)$ such that X is an Esakia space, and the following conditions hold:

- $R_\Box[x]$ is a closed upset
- $R_\Diamond[x]$ is a closed downset
- If U is a clopen upset, then $\Diamond_{R_\Diamond} U$ and $\Box_{R_\Box} U$ are clopen upsets

An **IK**-space is a $\Box\Diamond$-frame where $R := R_\Box \cap R_\Diamond$ and

(I) $R_\Diamond = \downarrow(R_\Box \cap R_\Diamond)$,

(II) $R_\Box = \leq \circ (R_\Box \cap R_\Diamond)$.

Coalgebras for $\Box\Diamond$-frames

Definition

The functors $\mathcal{V}^{\uparrow}(X)$ and $\mathcal{V}^{\downarrow}(X)$ (called the *upper Vietoris space* and *lower Vietoris space* of X) are defined as follows:

1. $\mathcal{V}^{\uparrow}(X) = (\{C \subseteq X | C \text{ is a closed upset}\}, \supseteq)$, with the topology given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen *upsets* of X;
2. $\mathcal{V}^{\downarrow}(X) = (\{C \subseteq X | C \text{ is a closed downset }\}, \subseteq)$, with the topology given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen downsets of X.

Where $[U] = \{C \in \mathcal{V}(X) | C \subseteq U\}$ and $\langle V \rangle = \{C \subseteq \mathcal{V}(X) | C \cap V \neq \emptyset\}$

Coalgebras for $\Box\Diamond$-frames

Theorem

$\Box\Diamond$-frames $(X, R_\Box, R_\Diamond)$ are in 1-1 correspondence with Priestley coalgebras $(X, \alpha : X \to \mathcal{V}^\uparrow(X) \times \mathcal{V}^\downarrow(X))$.

Then we have that the categories of $\Box\Diamond$-frames and **Coalg**$(\mathcal{V}_G(\mathcal{V}^\uparrow \times \mathcal{V}^\downarrow))$ are equivalent.

Algebraic correspondence: Let D_X be the dual distributive lattice to X. Then $\mathcal{V}^\uparrow$ dually corresponds to generating the free distributive lattice over $\{\Box a | a \in D_X\}$ and quotienting over the normality axioms for $\Box$. Similarly, $\mathcal{V}^\downarrow$ does so over $\{\Diamond a | a \in D_X\}$.

The functors FS_1 and FS_2

Now let's deal with the remaining axioms.

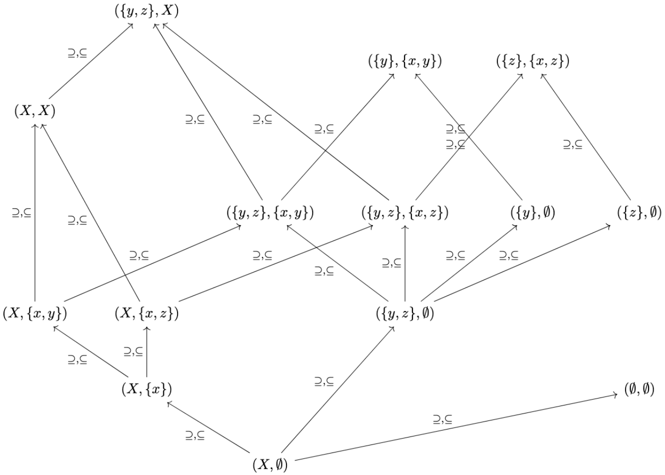
Definition

Let $FS_1(X) = \{(D, C) \in \mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X) : C = \downarrow (D \cap C)\}$.

Definition

Let $FS_2(X) = \{C \in V_r(FS_1(X)) | \forall (D, E) \in C$, $y \in D$ and $y \leq z$, there exists $(D', E') \geq (D, E)$ in C such that $z \in D' \cap E'\}$

Example: $FS_1(2F)$



The functors FS_1 and FS_2

Proposition

*$FS_1(X)$ is the Priestley subspace of $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$ such that axiom **A** dually holds.*

i.e.
$FS_1(X) = \{(D, C) \in \mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X) \mid \forall U, V \in ClopUp(X).\ (D, C) \in (\mathcal{V}^{\uparrow}(X) \times \langle U \to V \rangle) \cap ([U] \times \mathcal{V}^{\downarrow}(X)) \implies (D, C) \in \mathcal{V}^{\uparrow}(X) \times \langle V \rangle\}$.

Proposition

*FS_2 is the Priestley subspace of $V_r(FS_1)$ for which axiom **B** dually holds.*

i.e. $FS_2(X) = \{C \in \mathcal{V}_r(FS_1(X)) \mid \forall U, V \in ClopUp(X)\ .\ C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))] \implies C \in [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$.

Coalgebraic representation of Fischer-Servi logic

Theorem

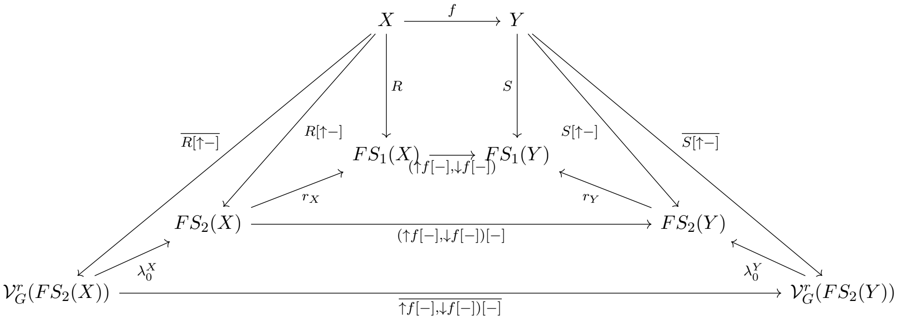
The following are in 1-1 correspondence:

1. ***IK**-spaces* (X, R)*,*
2. r*-open Priestley maps* $\alpha : X \to FS_2(X)$*, and*
3. *Coalgebras for the Esakia endofunctor* $\mathcal{V}_G^r(FS_2(X))$

Theorem

The category ***CoAlg***$(\mathcal{V}_G^r(FS_2(-)))$ *is equivalent to the category* ***IKS*** *of* ***IK****-spaces with p-morphisms.*

Lifting the p-morphisms



Commuting diagram for $\mathcal{V}_G^r(FS_2(-))$-coalgebras

Consequences: Bisimulation and free **IK**-algebras

- We obtain bisimulation for **IK** spaces directly from coalgebra bisimulations for $\mathcal{V}^r_G(FS_2(-))$.

Definition (Bisimulation for **IK** spaces)

Let (X, R) and (Y, S) be two **IK**-spaces. We say that a relation $\sim \subseteq X \times Y$ is a bisimulation provided

Forth:

1. Whenever $x \leq x'$ and $x \sim y$, there is some $y' \geq y$ such that $x' \sim y'$;
2. Whenever xRx' and $x \sim y$, there is some $y' \in S[y]$ such that $x' \sim y'$.

Back:

1. Whenever $y \leq y'$ and $x \sim y$, there is some $x' \geq x$ such that $x' \sim y'$;
2. Whenever ySy' and $x \sim y$, there is some $x' \in R[x]$ such that $x' \sim y'$.

- We derive a uniform construction of the free **IK**-algebra on any number of generators.

Image-Finite Posets

We derive analogous results for Kripke frames over image-finite posets using the functor P_G:

Definition

Let $g : X \to Y$ be a monotone map between image-finite posets. The *g-discrete complex* $(P^g_\bullet(X), \leq_\bullet)$ over X is a sequence

$$(P_0(X), P_1(X), ..., P_n(X), ...)$$

connected by morphisms $r_{i+1} : P_{i+1(X) \to P_i(X)}$ such that

- $P_0(X) = X$;
- $r_0 = g$;
- $P_{i+1}(X) := P_{r_i}(P_i(X))$
- $r_{i+1} := r_{r_i} : P_{i+1}(X) \to P_i(X)$ is the root map.

We denote the *image-finite part* of the projective limit of this family by $P_G(X)$.

Conclusions and Further Work

In this work we showed that the methods from Almeida and Bezhanishvili (2024) can be pushed to include Fischer-Servi logic.

One disadvantage of this method is that as seen, the functor V_G becomes infinite even if one only works with finite algebras. It would be interesting to work in the natural setting of intermediate logics where V_G preserves finiteness – like LC.

It would also be interesting to know what happens when one adds axioms with more complex modal depth.

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Thank you!

Free **IK** algebras

Definition

Let X be an Esakia space. Define the following sequence:

$$(M_0(X), M_1(X), ..., M_n(X), ...)$$

and a sequence of morphisms $\pi_k : M_k(X) \to M_{k-1}(X)$ for $k > 0$ and $\pi_0 : M_0(X) \to M_0(X)$ defined as follows:

- $M_0(X) = X$;
- $M_{n+1}(X) := X \times V_G^r(FS_2(M_n(X)))$
- $\pi_0 = id_{M_0}$ and $\pi_1(x, C) = x$;
- $\pi_{n+1}(x, C) = (x, (V_G^r(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_n)))(C)))$.

We denote the inverse limit (in **Pris**) of this system by $M_\infty(X)$.

Theorem

*Let X be a set of generators, and let $\mathbb{X}_{F_D(X)}$ denote the Priestley dual of the free distrbutive lattice $F_D(X)$ over X. Then $M_\infty(\mathbb{X}_{F_D(X)})$ is the dual to the free **FS**-algebra on X many generators.*