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HMS-Duality for Residuated Lattices

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Substructural Logics

Substructural Logics are logics often characterized by removing structural rules from the sequent calculus style proof systems for classical or intuitionsitic logic.

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Structural rules tell us how we can manipulate structures/premise-combinations without effecting their consequences.

$$(a)\ \frac{W[\Gamma; (\Delta; \Sigma)] \Rightarrow \varphi}{W[(\Gamma; \Delta); \Sigma] \Rightarrow \varphi} \qquad (w)\ \frac{W[\Gamma] \Rightarrow \varphi}{W[\Delta; \Gamma] \Rightarrow \varphi}$$

$$(e)\ \frac{W[\Gamma; \Delta] \Rightarrow \varphi}{W[\Delta; \Gamma] \Rightarrow \varphi} \qquad (c)\ \frac{W[\Gamma; \Gamma] \Rightarrow \varphi}{W[\Gamma] \Rightarrow \varphi}$$

Removing structural rules captures the idea that different organizations of the same information may lead to different consequences.

Additive Rules

$$\varphi := p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \bullet \varphi \mid \varphi \backslash \varphi \mid \varphi / \varphi \mid t.$$

Additive Rules:

$$(\wedge)\ \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi} \qquad (\wedge)\ \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}$$

$$(\vee)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\vee)\ \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

$$(\wedge\text{-in})\ \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \qquad (\vee\text{-out})\ \frac{\Gamma[\varphi] \Rightarrow \chi \qquad \Gamma[\psi] \Rightarrow \chi}{\Gamma[\varphi \vee \psi] \Rightarrow \chi}$$

$$(\text{Cut})\ \frac{\Gamma \Rightarrow \varphi \qquad Y[\varphi] \Rightarrow \psi}{Y[\Gamma] \Rightarrow \psi}$$

Multiplicative Rules

$$\varphi := p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \bullet \varphi \mid \varphi \backslash \varphi \mid \varphi / \varphi \mid t.$$

Multiplicative Rules:

$$(/\text{-in}) \ \frac{\Gamma; A \Rightarrow B}{\Gamma \Rightarrow B/A} \qquad (/\text{-out}) \ \frac{\Gamma \Rightarrow B/A \qquad Y \Rightarrow A}{\Gamma; \Delta \Rightarrow B}$$

$$(\backslash\text{-in}) \ \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \backslash B} \qquad (\backslash\text{-out}) \ \frac{\Gamma \Rightarrow A \backslash B \qquad \Delta \Rightarrow A}{\Delta; \Gamma \Rightarrow B}$$

$$(\bullet\text{-in}) \ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma; \Delta \Rightarrow A \bullet B} \qquad (\bullet\text{-out}) \ \frac{\Gamma \Rightarrow A \bullet B \qquad \Delta[A; B] \Rightarrow \xi}{\Delta[\Gamma] \Rightarrow \xi}$$

Substructural Logics 3

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The logic $\mathbf{NFL}$ is the least set of sequents containing all instances of the following sequents:

$$\varphi \Rightarrow \varphi \qquad \Gamma \Rightarrow \top \qquad \Gamma[\bot] \Rightarrow \varphi$$

$$t \bullet \varphi \Rightarrow \varphi \qquad \varphi \bullet t \Rightarrow \varphi \qquad \varphi \Rightarrow t \bullet \varphi \qquad \varphi \Rightarrow \varphi \bullet t$$

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and that is closed under the rules on the previous slides.

An **extension** $\mathbf{L}$ of $\mathbf{NFL}$ is a set of sequents containing $\mathbf{NFL}$ and closed under the rules of $\mathbf{NFL}$ and under substitutions.

Familiar Examples: Classical, Intuitionistic, Relevance, and Linear Logics.

Algebraic Semantics

The most extensively studied semantics for substructural logics is algebraic.

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$r\ell$-groupoids

A (bounded) $r\ell$-groupoid $\mathbf{G} = (G, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ is an algebra where $(G, \wedge, \vee, \top, \bot)$ is a lattice, $(G, \cdot, e, \leq)$ is a ordered groupoid with an identity element e, and $\cdot$, $\backslash$, and $/$ satisfy the residual law:

$$a \cdot b \leq c \iff b \leq a\backslash c \iff a \leq c/b.$$

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$\mathbf{G}$ is a *residuated lattice* when multiplication is associative.

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$$a \cdot b \leq c \iff b \leq a\backslash c \iff a \leq c/b.$$

A model $(\mathbf{G}, \sigma)$ consists of an $r\ell$-groupid and a valuation $\sigma : Prop \to G$.

We write $\mathbf{G} \vDash \Gamma \Rightarrow \varphi$ when for each $\sigma : Prop \to G$, $\sigma(\Gamma) \leq \sigma(\varphi)$.

Completeness of NFL w.r.t $r\ell$-groupoids

If $\mathbf{G} \vDash \Gamma \Rightarrow \varphi$ for all $r\ell$-groupoids $\mathbf{G}$, then $\Gamma \Rightarrow \varphi \in \mathbf{NFL}$.

OKHD-Semantics

Less well known is the operational/Kripke semantics pioneered by Ono and Komori [1985], Humberstone [1987], and Došen [1989].

Their key insight is the treatment of disjunction!

Allows for completeness of non-distributive logics.

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OKHD-frames

An *OKHD-frame* is a structure $(X, \curlywedge, 1, \otimes, \varepsilon)$ where $(X, \curlywedge, 1)$ is a **meet semi-lattice**, $\otimes$ is a **binary operation** on X, ε is **an identity element** for $\otimes$, and the following identities hold.

$$x \otimes 1 = 1 = 1 \otimes x$$

$$x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z) \qquad (y \curlywedge z) \otimes x = (y \otimes x) \curlywedge (z \otimes x)$$

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We then define a satisfaction relation:

$x \Vdash p$ iff $x \in V(p)$

$x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$

$x \Vdash \varphi \vee \psi$ iff there are y, z such that $y \curlywedge z \leq x$ and $y \Vdash \varphi$ and $y \Vdash \psi$

$x \Vdash \varphi \bullet \psi$ iff there are y, z such that $y \otimes z \leq x$ and $y \Vdash \varphi$ and $y \Vdash \psi$

$x \Vdash \varphi \backslash \psi$ iff forall y, if $y \Vdash \varphi$, then $y \otimes x \Vdash \psi$

$x \Vdash \psi / \varphi$ iff for all y, if $y \Vdash \varphi$, then $x \otimes y \Vdash \psi$

$x \Vdash t$ iff $\varepsilon \leq x$ $\qquad$ $x \Vdash \top$ iff $x \in X$ $\qquad$ $x \Vdash \bot$ iff $x = 1$.

$x \Vdash \Gamma; \Delta$ iff there are y, z such that $y \otimes z \leq x$ and $y \Vdash \Gamma$ and $z \Vdash \Delta$.

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$x \Vdash \Gamma; \Delta$ iff there are y, z such that $y \otimes z \leq x$ and $y \Vdash \Gamma$ and $z \Vdash \Delta$.

Given a model $\mathbf{M}$, we write $[\![\varphi]\!]$ for the set of points in $\mathbf{M}$ satisfying φ.

Persistence

For all φ and all models $\mathbf{M} = (\mathbf{X}, V)$, $[\![\varphi]\!]$ is a filter of $\mathbf{X}$.

OKHD-Semantics

We write $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$ iff for all $V : Prop \to \mathcal{F}i(X)$, $[\![\Gamma]\!] \subseteq [\![\varphi]\!]$.

Completeness of $\mathbf{NFL}$ w.r.t OKHD-frames

If $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$ for all OKHD-frames $\mathbf{X}$, then $\Gamma \Rightarrow \varphi \in \mathbf{NFL}$.

Originally proved for $\mathbf{NFL}$ via a canonical model style proof Došen [1989].

Also obtained completeness for a number of extensions and recovered the completeness proofs of Ono and Komori [1985] and Humberstone [1987].

Connecting Algebraic Semantics to the OKHD-semantics

The algebraic semantics and OKHD-semantics are connected in so far as $Log(\mathbf{OKHD})$ is the same as $Log(\mathbf{RLG})$.

How is this connection characterized without directly appealing to their common logic?

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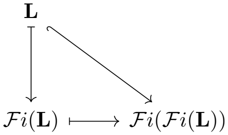
Ono and Komori: for the algebras of logics without contraction.

$$\mathbf{L} \mapsto \mathcal{F}i(\mathbf{L}) \qquad\qquad \mathbf{X} \mapsto \mathcal{F}i(\mathbf{X})$$



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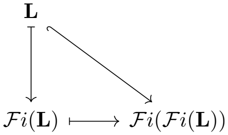

Yields embedding theorem.

Does not yield a isomorphism in general.

Loss of information: some classes of algebras not closed under the composition of the operations.

Topological Duality in Logic

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Provide us with both methodological and conceptual insight.

Methodological:

- Topological Semantics
  - All normal modal logics are topologically complete via duality,
  - Straight forward connection to Kripke semantics.

$$\begin{array}{ccc} \mathbf{MA}^{op} & \longleftrightarrow & \mathbf{MSp} \\ & & \downarrow \\ & & \mathbf{KR} \end{array}$$

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Philosophical:

- propositions as primitive entities and worlds as sets of propositions vs.
- worlds as primitive entities and propositions as sets of worlds,

In light of duality, neither perspective is prior to the other.

Modal Logic, Duality, and Canonicity

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Canonicity via d-persistence

$$\begin{array}{ccc} \vdash_{\mathbf{L}} \varphi & \longrightarrow & \Lambda_{\mathbf{L}} \vDash \varphi \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\mathbf{L}} \vDash \varphi & \longleftarrow & X_{\Lambda_{\mathbf{L}}} \vDash \varphi \end{array}$$

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Sahlqvist: every formula of the right shape is d-persistent and thus canonical.

Duality for $r\ell$-Groupoids

Is there a topological duality a for $r\ell$-groupoids that maintains a connection to the OKHD-semantics.

Is there a notion of persistence that can allow us to obtain general canonicity/completeness proofs with respect to OKHD-frames?

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Yes!

Strategy:

- Modify a recent duality for not-necessarily-distributive lattices introduced by Bezhanishvili et al. [2024].
  - Bezhanishvili et al. [2024] restrict HMS-Duality for semilattices.
- Use their analogue of d-persistence, called Π_1-persistence, to our setting.

NRL-Spaces

NRL-Spaces

An NRL-space $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ is a **compact** topological **semilattice** $(X, \curlywedge, 1, \tau)$ with a **basis of clopens** equipped with a **binary operation** $\otimes$ and an **identity element** ε for $\otimes$ satisfying (1)-(4):

1) If $x \not\leq y$, then there is a clopen filter U such that $x \in U$ and $y \notin U$,
2) If U, V are clopen filters, then so are

$$U \triangledown V = \{x \curlywedge y \mid x \in U \ \& \ y \in V\}$$
$$U \circ V = \{x \otimes y \mid x \in U \ \& \ y \in V\}$$
$$U \backslash V = \{y \mid \forall x (x \in U \to x \otimes y \in V)\}$$
$$V / U = \{x \mid \forall y (y \in U \to x \otimes y \in V)\},$$

3) $x \otimes y \leq z$ iff for all clopen filters U, V: if $x \in U$ and $y \in V$, then $z \in U \circ V$,
4) ε and $\{1\}$ are clopen.

L-Spaces are compact topological semilattices with a basis of clopens satisfying (1) and that the clopen filters contain $\{1\}$ and are closed under $\triangledown$.

Duality for $r\ell$-Groupoids and NRL-Spaces

There are also well defined morphisms between NRL-spaces.

Theorem

The category of $r\ell$-groupoids is dually equivalent to the category of NRL-spaces.

Proof:
There are functors:

$$\mathbf{G} \mapsto \mathbf{X_G} = (\mathcal{F}i(G), \cap, G, \otimes, e, \tau) \qquad \mathbf{X} \mapsto \mathbf{G_X} = (\mathcal{F}i_{clp}(X), \cap, \triangledown, X, 1, \circ, \backslash, /, \varepsilon).$$

$$f \mapsto f^{-1} \qquad\qquad g \mapsto g^{-1}$$

Here $x \otimes y = \{a \cdot b \mid a \in x \;\&\; b \in y\}$ and τ is generated by the subbase $\{\phi(a) \mid a \in L\} \cup \{\phi(a)^c \mid a \in L\}$ where $\phi(a) = \{x \in \mathcal{F}i(L) \mid a \in x\}$.

$\phi : \mathbf{G} \to \mathbf{G_{X_G}}$ and $\eta : \mathbf{X} \to \mathbf{X_{G_X}}$ defined by $\phi(a) = \{x \in \mathcal{F}i(G) \mid a \in x\}$ and $\eta(x) = \{U \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in U\}$ are both isomorphisms.

Restrictions

We can can explicitly characterize the dual spaces of residuated lattices, FL-spaces, and involutive residuated lattices.

$$\begin{array}{ccccccc}
\mathbf{NRLSp} & \longleftarrow & \mathbf{RLSp} & \longleftarrow & \mathbf{FLSp} & \longleftarrow & \mathbf{InFLSp} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathbf{RLG}^{op} & \longleftarrow & \mathbf{RL}^{op} & \longleftarrow & \mathbf{FL}^{op} & \longleftarrow & \mathbf{InRL}^{op}
\end{array}$$

FL-algebras are RLs with an additional distinguished element, f. We can define two negations $\sim a := a \backslash f$ and $\neg a := f/a$. We say that an FL-algebra is an involutive RL if $\sim\neg a = a$ and $\neg\sim a = a$.

(Structured) Topological Completeness

A topological model is a pair $(\mathbf{X}, V)$ where $\mathbf{X}$ is an NRL-space and $V : Prop \to \mathcal{F}i_{clp}(\mathbf{X})$ is a clopen valuation.

Satisfaction is defined the same as for OKHD-semantics.

Theorem: Topological Completeness

Every extension $\mathbf{L}$ of $\mathbf{NFL}$ is sound and complete with respect to a class of NRL-spaces.

Getting back to OKHD-frames

Theorem: NRL to OKHD

If $\mathbf{X}$ is an NRL-space, then the following identities hold.

$$x \otimes 1 = 1 = 1 \otimes x$$

$$x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z) \qquad (y \curlywedge z) \otimes x = (y \otimes x) \curlywedge (z \otimes x)$$

So $\mathbf{X}$ is an OKHD-frame.

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So $\mathbf{X}$ is an OKHD-frame.

A special case:

Canonical Model and Lindenbaum Algebra

For each extension $\mathbf{L}$ of $\mathbf{NFL}$ let $\Lambda_{\mathbf{L}}$ be the Lindenbaum algebra of $\mathbf{L}$, then:

The topology free reduct of $\mathbf{X}_{\Lambda_{\mathbf{L}}}$ is isomorphic to the canonical frame of $\mathbf{L}$.

Canonical frame defined à la Došen [1989].

Π_1-persistence and OKHD-Canonicity

Π_1-persistence: If $(\mathbf{X}, \tau) \vDash \Gamma \Rightarrow \varphi$, then $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$.

Theorem: Π_1-Persistence

Every sequent $\Gamma \Rightarrow \varphi$ in the signature $\top, \bot, t, \wedge, \vee, \bullet$ is Π_1-persistent.

Π_1-persistence and OKHD-Canonicity

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Theorem: Π_1-Persistence

Every sequent $\Gamma \Rightarrow \varphi$ in the signature $\top, \bot, t, \wedge, \vee, \bullet$ is Π_1-persistent.

OKHD-Canonicity: Whenever $\Gamma \Rightarrow \varphi \in \mathbf{L}$, then $\mathfrak{X}_{\mathbf{L}} \vDash \Gamma \Rightarrow \varphi$.

Corollary: OKHD-Canonicity

Every extension of $\mathbf{NFL}$ axiomatized by sequents in the signature $\top, \bot, t, \wedge, \vee, \bullet$ is OKHD-canonical.

$$\begin{array}{ccc} \Gamma \Rightarrow \varphi \in \mathbf{L} & \longrightarrow & \Lambda_{\mathbf{L}} \vDash \Gamma \Rightarrow \varphi \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\mathbf{L}} \vDash \Gamma \Rightarrow \varphi & \longleftarrow & X_{\Lambda_{\mathbf{L}}} \vDash \Gamma \Rightarrow \varphi \end{array}$$

Conclusion

- Developed a duality connecting the algebraic semantics of NFL to the OKHD-semantics.

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- Showed how this duality can be used to generalize existing completeness theorems and canonicity results.

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Thank You :)

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