Group Epistemics, (Co-)algebraically

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CELIA - *Coalition and Epistemic Logic: An Intensional Approach to Groups*

Epistemic Logics of Structured Intensional Groups: Agents - Groups - Names - Types

- Epistemic logics provide logical approaches to knowledge, belief and related attitudes assigned to (human or artificial) agents.
- The theory of knowledge is in particular applied in distributed systems to the design of efficient fault-tolerant protocols.
- Various notions of group knowledge: somebody knows, everybody knows, common knowledge, distributed knowledge.
- Common knowledge is a prerequisite of coordinated, simultaneous action in synchronous systems.
- But how do we determine a group? By its members (players of a game)? What if membership changes or is uncertain (dead/alive processes, trolls, liberals)? Or definition of a (sub)group is not rigid (consistent subgroups)?
- Lift the assumption that groups are rigidly determined by membership and that membership in a group is commonly known by the agents.
- What changes on the logical side of things?

Knowledge of (groups of) agents

Epistemic models and logic

$(W, A, \{\sim_a \mid a \in A\})$ where A is a fixed finite set of agents

- CPC interpreted over the BA on $\mathcal{P}(W)$
- $w \Vdash \Box_a \varphi \;\equiv\; \forall u(u \sim_a w \longrightarrow u \Vdash \varphi)$

= modal logic S5

Group knowledge

- $E_G\varphi \;\equiv\; \bigwedge_{a\in G} \Box_a \varphi$ $\qquad \bigcup_{a\in G} \sim_a$
- $C_G\varphi \;\equiv\; \bigwedge_{k\in N} (E_G)^n \varphi$ $\qquad (\bigcup_{a\in G} \sim_a)^*$
- $D_G\varphi$ $\qquad \bigcap_{a\in G} \sim_a$

Interpretation of groups is extensional (a fixed set of its members) and rigid.

Related work

- Non-rigid knowledge for groups with names (Grove & Halpern 1993)
- Non-rigid common and distributed knowledge (Moses & Tuttle 1988; MB, Christoff & Roy 2021)
- Rigid common knowledge for formula-defined groups (Humml & Schröder 2023)
- Term-modal logics (Naumov & Tao 2018, Wang & Selingman 2018)
- Non-rigid knowledge for groups with algebraic labels (MB & Sedlár 2023)

Knowledge for groups with algebraic labels

Language

Let Σ be an algebraic similarity type, and Pr, Gr denumerable sets of propositional variables and group variables. The sets of Σ-terms and Σ-formulas:

$$Tm_\Sigma : \alpha := \mathsf{a} \in Gr \mid o(\alpha_1, \ldots, \alpha_n) \text{ for } o \in \Sigma$$

$$Fm_\Sigma : \varphi := \mathsf{p} \in Pr \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\alpha]\varphi \mid \langle\alpha]\varphi.$$

$[\alpha]\varphi$ read as "Everyone in the group (given by) α knows that φ", and
$\langle\alpha]\varphi$ read as "Someone in the group (given by) α knows that φ".

Example: Join-Semilattices

$\Sigma_{\mathsf{SL}} = \{+, 0\}$ $\quad [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$

Example: Boolean algebras (groups defined by formulas)

Σ_{BA} $\quad \langle\top]\varphi \leftrightarrow \langle\neg\alpha]\varphi \vee \langle\alpha]\varphi$

Knowledge for groups with algebraic labels

Neighborhood Σ-frames

A neighborhood Σ-frame $\mathfrak{F}$ is a tuple $(W, \mathbf{G}, \{\nu_a\}_{a\in G})$ where

- W is a set of states,
- $\mathbf{G}$ is a Σ-algebra,
- for each $a \in G$, $\nu_a : W \longrightarrow \mathcal{P}\mathcal{P}(W)$ assigns to each w a set of sets of states (neighborhoods).
  Core nbhd sets $X \in \nu_a(w)$ represent agents in a, as seen from w.

The complex algebra $\mathfrak{F}^+$ is given as the expansion of the boolean algebra of subsets of W by

$$[a]^+P = \{w \mid \forall X \in \nu_a(w) : X \subseteq P\} \qquad \langle a]^+P = \{w \mid \exists X \in \nu_a(w) : X \subseteq P\}.$$

$$\Box_a^+P = \{w \mid P \in \nu_a(w)\} \qquad \langle a)^+P = \{w \mid \exists \mathcal{X} \subseteq \nu_a(w) : \mathcal{X} \neq \emptyset \;\&\; \cap \mathcal{X} \subseteq P\}.$$

Knowledge for groups with algebraic labels

The minimal logic of $[\alpha]$, $\langle\alpha]$ (for $\Sigma = \emptyset$)

1. substitution instances of classical tautologies and is closed under MP;
2. (K) $[\alpha](\varphi \longrightarrow \psi) \longrightarrow ([\alpha]\varphi \longrightarrow [\alpha]\psi)$
3. (Nec) $\dfrac{\varphi}{[\alpha]\varphi}$
4. (Int_1) $\neg[\alpha]\bot \longrightarrow \langle\alpha]\top$
5. (Int_2) $\langle\alpha]\varphi \wedge [\alpha]\psi \longrightarrow \langle\alpha](\varphi \wedge \psi)$.

The minimal logic does not assume any algebraic properties for the assignment $a \mapsto \nu_a : W \longrightarrow \mathcal{P}\mathcal{P}(W)$. It is complete w.r.t. the nbhd (or dually, algebraic) semantics. Additional properties of the assignment $\nu_a : W \longrightarrow \mathcal{P}\mathcal{P}(W)$ yield extensions of the minimal logics.

The minimal logic coincides with multilabel bi-modal monotone logic biM (Brown: On the logic of ability, JPL 1988). It is a combination of K with M, with interacting modalities (given by the common core nbhd assignment).

Adding common or distributed knowledge

Example: $\Sigma = \emptyset$ (Epistemic logic with names $n \in \mathbb{N}$)

Common knowledge is given in terms of $[n]$: $C_n\varphi := \bigwedge_{k\in\mathbb{N}} [n]^k\varphi$.

$$
\begin{array}{ll}
\mathrm{K(C_n)} & C_n(\varphi \to \psi) \to (C_n\varphi \to C_n\psi) \\
\mathrm{FP} & C_n\varphi \to [n](\varphi \wedge C_n\varphi) \\
\mathrm{Ind} & \textit{From } \varphi \to [n](\varphi \wedge \psi), \textit{ infer } \varphi \to C_n\psi \\
\mathrm{Nec(C_n)} & \text{From } \varphi, \text{ infer } C_n\varphi
\end{array}
$$

Distributed knowledge is given in terms of $\langle n)$:

$$
\begin{array}{ll}
\mathrm{K(\langle n))} & \langle n)\varphi \wedge \langle n)(\varphi \to \psi) \to \langle n)\psi \\
\mathrm{Incl} & \langle n]\varphi \to \langle n)\varphi \\
\mathrm{Int_D} & \langle n)\varphi \wedge [n](\varphi \to \psi) \to \langle n)\psi
\end{array}
$$

Marta Bílková, Zoé Christoff & Olivier Roy (2021): Revisiting Epistemic Logic with Names. TARK 2021, pp. 39–54.

Morphisms and bisimulations

Morphisms $(W, \mathbf{G}, \nu) \longrightarrow (W', \mathbf{G}', \nu')$: pairs of maps $(g : \mathbf{G} \longrightarrow \mathbf{G}', f : W \longrightarrow W')$

Neighborhood Σ-frame core-morphisms (PP-coalgebras)

(there) $X \in \nu_a(w) \Rightarrow f[X] \in \nu'_{g(a)}(f(w))$

(back) $Y \in \nu'_{g(a)}(f(w)) \Rightarrow \exists X(f[X] = Y \;\&\; X \in \nu_a(w))$

General neighborhood Σ-frame morphisms (N-coalgebras)

$$f^{-1}[Y] \in \nu_a(w) \;\equiv\; Y \in \nu'_{g(a)}(f(w))$$

Neighborhood Σ-frame morphisms (MxF-coalgebras)

(M-there) $X \in \nu_a(w) \Rightarrow \exists Y(Y \subseteq f[X] \;\&\; Y \in \nu'_{g(a)}(f(w)))$

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(F) $\cup \nu'_{g(a)}(f(w)) = \cup\{f[X] \mid X \in \nu_a(w))\}$

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Core-morphisms preserve validity ($\langle a]$, $[a]$, $\Box_a$):
Let $(f, g) : (W_1, \mathbf{G_1}, \{\nu_a\}_{a \in G_1}) \longrightarrow (W_2, \mathbf{G_2}, \{\nu_a\}_{a \in G_2})$ be a neighborhood Σ-frame s-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. Then for each formula φ and each $w \in W$,

$$\mathfrak{F}_1, w \Vdash \varphi \quad \Rightarrow \quad \mathfrak{F}_2, f(w) \Vdash \varphi.$$

M. Bílková & I. Sedlár (2023): Epistemic Logics of Structured Intensional Groups. TARK 2023.

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A core-bisimulation of neighborhood Σ-models $(W_1, \nu^1, \mathbf{G_1}, [\![]\!]_1)$ and $(W_2, \nu^2, \mathbf{G_2}, [\![]\!]_2)$ is a pair $(\cong, B)$, with $\cong \subseteq \mathbf{G_1} \times \mathbf{G_2}$ a congruence, and $B \subseteq W_1 \times W_2$, s.t.

$$\forall \mathrm{a} \in Gr\ [\![\mathrm{a}]\!]_1 \cong [\![\mathrm{a}]\!]_2 \text{ and } \forall \mathrm{p} \in Pr\ (w_1 B w_2 \Rightarrow (w_1 \in [\![\mathrm{p}]\!]_1 \Leftrightarrow w_2 \in [\![\mathrm{p}]\!]_2))$$

$$(w_1 B w_2 \wedge a_1 \cong a_2) \Rightarrow (\forall X \in \nu^1_{a_1}(w_1)\ \exists Y \in \nu^2_{a_2}(w_2)\ X\ \overline{B}\ Y)$$

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Bisimilarity implies modal equivalence ($\langle a]$, $[a]$). Graphs of neighborhood Σ-frame core-morphisms are examples of core-bisimulations, and functional core-bisimulations correspond to graphs of neighborhood Σ-frame core-morphisms. H.M. theorem fails in its standard formulation.

Morphisms and bisimulations

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- Algebraic dual adjunction ($\langle a]$, $[a]$):
  Two-sorted algebras $\mathfrak{A} = (\mathbf{B}, \mathbf{G}, [\,], \langle\,])$, where $\mathbf{B} = (X, \wedge, \vee, \neg, \top, \bot)$ is a Boolean algebra; $\mathbf{G}$ is a Σ-type algebra; and $[\,]$ and $\langle\,]$ are functions of the type $\mathbf{B} \times \mathbf{G} \longrightarrow \mathbf{B}$ such that

$$[a]\top = \top \qquad [a](x \wedge y) = [a]x \wedge [a]y$$
$$\neg[a]\bot \leq \langle a]\top \qquad \langle a]x \wedge [a]y \leq \langle a](x \wedge y) .$$

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Definability:
For the Σ_{SL}-neighborhood frames,

$$\langle \alpha + \beta] \varphi \leftrightarrow \langle \alpha] \varphi \vee \langle \beta] \varphi \qquad [\alpha + \beta] \varphi \leftrightarrow [\alpha] \varphi \wedge [\beta] \varphi$$

corresponds to properties:

$$\forall w (\uparrow \nu_{a+b}(w) = \uparrow \nu_a(w) \cup \uparrow \nu_b(w)) \quad \forall w (\uparrow\{\cup \nu_{a+b}(w)\} = \uparrow\{\cup \nu_a(w)\} \cap \uparrow\{\cup \nu_b(w)\})$$

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Definability: For the $\emptyset$-neighborhood frames (names), the set of formulas

$$\{\langle n] p \rightarrow p \mid n \in N\}$$

defines the class of frames satisfying

$$\forall w \forall n \; (X \in \nu_n(w) \longrightarrow w \in X).$$

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Definability: By the standard argument based on the (two-sorted) duality and Birkhoff's theorem:

Theorem

Let K be a class of neighborhood Σ-frames closed under taking ultrafilter extensions. Then K is definable iff K is closed under disjoint unions, generated subframes and bounded morphic images, and reflects ultrafilter extensions.

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- A bisimulation of neighborhood Σ-models $(W_1, \nu^1, \mathbf{G_1}, [\![]\!]_1)$ and $(W_2, \nu^2, \mathbf{G_2}, [\![]\!]_2)$ is a pair $(\cong, B)$, with $\cong \subseteq \mathbf{G_1} \times \mathbf{G_2}$ a congruence, and $B \subseteq W_1 \times W_2$, s.t.

  $\forall \mathrm{a} \in Gr\ [\![\mathrm{a}]\!]_1 \cong [\![\mathrm{a}]\!]_2$ and $\forall \mathrm{p} \in Pr\ (w_1 B w_2 \Rightarrow (w_1 \in [\![\mathrm{p}]\!]_1 \Leftrightarrow w_2 \in [\![\mathrm{p}]\!]_2))$

  $(w_1 B w_2 \wedge a_1 \cong a_2) \Rightarrow (\forall X \in \nu^1_{a_1}(w_1)\ \exists Y \in \nu^2_{a_2}(w_2)\ X\ \overleftarrow{B}\ Y)$

  $(w_1 B w_2 \wedge a_1 \cong a_2) \Rightarrow (\forall Y \in \nu^2_{a_2}(w_2)\ \exists X \in \nu^1_{a_1}(w_1)\ X\ \overrightarrow{B}\ Y)$

  $\bigcup \nu^1_{a_1}(w_1)\ \overline{B}\ \bigcup \nu^2_{a_2}(w_2)$

  Bisimilarity implies modal equivalence ($\langle a]$, $[a]$). H.M. theorem holds in its standard formulation for image finite models.

Coalgebras

Starting with (multilabel) biM:

- biM combines a monotone and a normal modality,
- the $\langle\alpha]$ and $[\alpha]$ modalities are neighborhood modalities for monotone and augmented neighborhood frames respectively

We propose a setting making all of the above visible. It involves the following functors (monads) in Set:

- neighborhood monad $NX = [[X, 2], 2]$,
- monotone neighborhood monad M
- filter monad F

Multilabel setting (with set of labels A) would use N^A, M^A, F^A.
With Σ-algebra of terms, we seek an additional natural Σ-algebra on the functor in question:
Hansen, H.H., Kupke, C., Leal, R.A. (2014). Strong Completeness for Iteration-Free Coalgebraic Dynamic Logics. TCS 2014.

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With Σ-algebra of terms, it is sufficient to seek operations on the coalgebras (not the functor itself), so that reducible operations are safe for bisimulation:
H. H. Hansen and W. Poiger. Safety and Strong Completeness via Reducibility for Many-Valued Coalgebraic Dynamic Logics. (CALCO 2025).

Coalgebras

The functor:

$$M\hat{}F \subseteq M \times F$$

where $M\hat{}F X = \{(\mathtt{M}, \mathtt{F}) \in MX \times FX \mid \exists \mathtt{N} \in NX.\ \mathtt{M} = {\uparrow}N \ \ \&\ \ {\uparrow}\{\cup \mathtt{N}\} = \mathtt{F}\}$

A $M\hat{}F$-coalgebra

$$c : X \longrightarrow M\hat{}F X$$

can be perceived as a span of two coalgebras having a common generating core

$$\begin{array}{ccccc} MX & \xleftarrow{\ \uparrow .\ } & NX & \xrightarrow{\ \uparrow\{\cup .\}\ } & FX \\ & {\scriptstyle \mu}\nwarrow & \uparrow{\scriptstyle \nu} & \nearrow{\scriptstyle \varphi} & \\ & & X & & \end{array}$$

$M\hat{}F$-coalgebra morphisms = nbhd frame morphisms

$$\begin{array}{ccc} X & \xrightarrow{\ c\ } & M\hat{}F X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M\hat{}F f} \\ Y & \xrightarrow[\ d\ ]{} & M\hat{}F Y \end{array}$$

Coalgebras

The functor:

$$M\hat{\ }F \subseteq M \times F$$

where $M\hat{\ }FX = \{(\mathtt{M}, \mathtt{F}) \in MX \times FX \mid \exists \mathtt{N} \in NX.\ \mathtt{M} = \uparrow N \ \ \&\ \ \uparrow\{\cup \mathtt{N}\} = \mathtt{F}\}$

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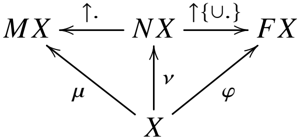
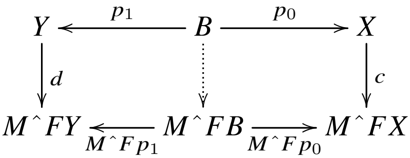
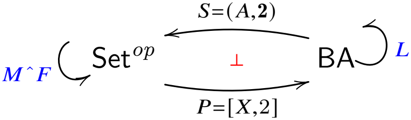


$M\hat{\ }F$-coalgebra bisimulations = nbhd frame bisimulations



The minimal logic, coalgebraically

The dual adjunction:



together with natural transformation $\delta : LP \longrightarrow P(M\hat{}F)$, which for a coalgebra $c : X \longrightarrow M\hat{}FX$, assigns L-algebra:

$$\hat{P}c = LPX \xrightarrow{\delta_X} P(M\hat{}F)X \xrightarrow{Pc} PX\,.$$

The semantics is given by (the unique arrow from) the initial L-algebra to $\hat{P}c$, mapping a formula to its extension.
L adds one layer of modalities, modulo the axioms, including the two interaction axioms.

Functor PP

- not a monad, but allows for a natural notion of multiplication, safe for bisimulations
- a composition of groups (= groups reflecting on other groups) with reduction axioms: $[\alpha \cdot \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$, $\langle \alpha \cdot \beta]\varphi \leftrightarrow \langle \alpha]\langle \beta]\varphi$ (e.g. Σ of right unital magmas)
- language not expressive, but a fragment of expressive Instantial nbhd logic

J. van Benthem, N. Bezhanishvili, S. Enqvist. A PDL for Instantial neighborhood semantics, Studia Logica 2018.

Functor $M\hat{\ }F$

- a subfunctor of a monad $M \times F$ (not likely a submonad)
- a sub-natural (lax natural) transformation to N allows for a sub-natural (lax natural) notion of multiplication
- only weakly associative composition of groups (= groups reflecting on other groups): $(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$

Work in progress and further connections

General theory

- Completeness of the algebraically labeled biM (cf. H. Hansen, C. Kupke, R. Leal 2014)
- Topological duality (cf. Guram Bezhanishvili, Nick Bezhanishvili, Jim de Groot, LMCS 2022)
- Interpolation properties of the minimal logic, and with labels

Common and distributive knowledge for intensional groups

- Common knowledge expressed as a fixpoint using $[\alpha]$ modality: proof theory and complexity
- Distributed knowledge: generalize from names to algebraic types, proof theory and complexity
- Common and distributive knowledge as genuine types of knowledge using appropriate algebraic signatures

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