Difference–restriction algebras with operators

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Algebraic study of partial functions

(Partial) functions important throughout formal sciences!

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1. Choose some operations: $\cap, \cup, {}^{\mathrm{c}}$
2. Any collection of subsets closed under operations an algebra
3. Algebras validate laws: $A \cap B = B \cap A \ldots$
4. Class of isomorphs of algebras: **BA**

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Use algebra to study partial functions

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2. Any collection of partial functions closed under operations an algebra
3. Algebras validate laws
4. Class of isomorphs of algebras: **DRA**

The project

Stone duality:

$$\mathbf{BA} \leftrightarrows \mathbf{Stone}^{\mathrm{op}}$$

Jónsson and Tarski: Boolean algebras with operators

$$\mathbf{BAO} \leftrightarrows \mathsf{DescriptiveGeneralFrame}^{\mathrm{op}}$$

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Same for algebras of partial functions:

'base':

$$\mathbf{DRA} \leftrightarrows ?^{\mathrm{op}}$$

Difference–restriction algebras with operators

$$\mathbf{DRAO} \leftrightarrows ?^{\mathrm{op}}$$

I: The algebras

Operations on partial fuctions

Definition

Partial function $f : X \rightharpoonup Y$ $\quad f \subseteq X \times Y$ s.t.

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- **Domain restriction:**

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- Intersection:

$$f \cap g = f - (f - g)$$

Algebras

Definition

Let $\mathcal{A} = (A, -, \triangleright)$. Call $\mathcal{A}$ an **algebra of partial functions** if:

$$\exists X, Y \quad A \subseteq (X \rightharpoonup Y)$$

and operations $-$ and $\triangleright$ are interpreted as:

- $-$: set difference
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Properties:

$$a \leq b \iff a \triangleright b = a$$

$$a \leq b \iff a \cdot b = a \quad \text{(same)}$$

meet semilattice

Class of difference–restriction algebras

Axioms

$$a - (b - a) = a \tag{1}$$

$$a \cdot b = b \cdot a \tag{2}$$

$$(a - b) - c = (a - c) - b \tag{3}$$

$$(a \triangleright c) \cdot (b \triangleright c) = (a \triangleright b) \triangleright c \tag{4}$$

$$(a \cdot b) \triangleright a = a \cdot b \tag{5}$$

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Category **DRA**

objects: difference–restriction algebras
morphisms: homomorphisms of $\{-, \triangleright\}$-algebras

Difference–restriction algebras: Properties

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$$a^{\downarrow} = \{x \mid x \leq a\} \quad \text{BA}$$

$$b, c \in a^{\downarrow} \quad \text{meet } b \cdot c \ / \ b \rhd c$$

$$\overline{b} = a - b$$

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Definition

compatible: $a \rhd b = b \rhd a$

…but join does not have to exist!

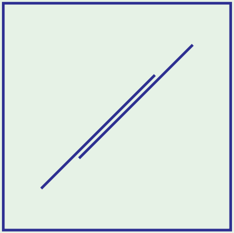
Example difference–restriction algebras

Example (1)

Identity functions on X

$\cdot$ and $\rhd$ coincide

(BA)



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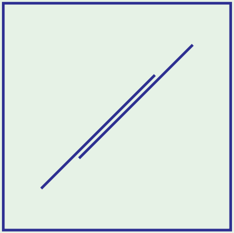

Example (2)

Identity functions on X with finite domain

(Generalised BA)

Example difference–restriction algebras

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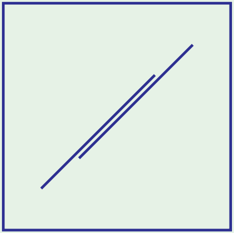

Example (2)

Identity functions on X with finite domain

(Generalised BA)

Example (3)

Identity functions on X with $|\mathrm{dom}| \leq 10$

II: From algebras to spaces

Adjunction

F : **DRA** $\dashv$ $?^{\mathrm{op}}$ (topological)

Prime filters

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- upward closed
- non-empty
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$\iff$ maximal proper

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μ, ν prime

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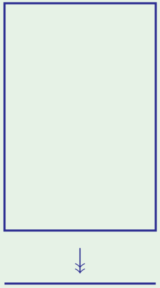
- $F(\mathcal{A}) =$

$$\pi_{\mathcal{A}} \colon (\mathrm{Prime}(A), \tau_{\mathcal{A}}) \twoheadrightarrow (\mathrm{Prime}(\mathcal{A})/{\approx}, \tau_{\mathcal{A}}/{\approx})$$

Example: functor F on objects

Example

- X, Y finite $\quad |X| = |Y| = n$
- $|X \rightharpoonup Y| = (n+1)^n = 2^{n \log(n+1)}$
- μ prime: $\quad \mu = \{f \mid (x, y) \in f\}$ some $(x, y) \in X \times Y$
- $\pi : X \times Y \twoheadrightarrow X$
  $(x, y) \mapsto x$
- $|\mathrm{Prime}| = n^2$



III: The spaces

Hausdorff étale spaces

spaces X, X_0

Definition

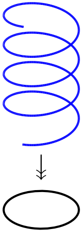
Local homeomorphism: $\pi \colon X \to X_0$

$\forall x \in X$ $\exists$ open neighbourhood $U \ni x$

- $\pi(U)$ open
- $\pi \colon U \to \pi(U)$ homeomorphism

Étale space: $\pi \colon X \twoheadrightarrow X_0$ surjective local homeomorphism

Hausdorff étale space: X (thus X_0) Hausdorff



Category of Hausdorff étale spaces

Category **HausEt**

object: Hausdorff étale space

morphism:

- continuous
- proper

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

- preserves equivalence
- fibrewise injective
- fibrewise surjective

IV: From spaces to algebras

Functor G on objects

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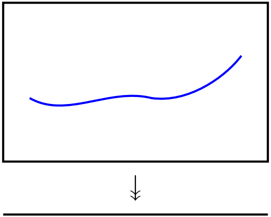
partial section $f \subseteq X$

$x, x' \in f$

$\pi(x) = \pi(x') \implies x = x'$



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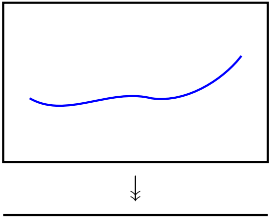

G: compact open partial sections

Main result: Adjunction

$$F\colon \mathbf{DRA} \leftrightarrows \mathbf{HausEt}^{\mathrm{op}} : G$$

F and G on morphisms : "inverse image"

Theorem (Borlido, Kudryavtseva, McLean)

The functors $F\colon \mathbf{DRA} \to \mathbf{HausEt}^{\mathrm{op}}$ *and* $G\colon \mathbf{HausEt}^{\mathrm{op}} \to \mathbf{DRA}$ *form an adjunction.*

That is

$$F \dashv G$$

Finite compatible completions

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1. $G(F(\mathcal{A}))$ has joins for each finite pairwise compatible subset
   finitarily compatibly complete
2. $\mathcal{A} \hookrightarrow G(F(\mathcal{A}))$
   every element of $G(F(\mathcal{A}))$ join of a finite compatible subset of (image) of $\mathcal{A}$
   finite join dense

$1 + 2$: $G(F(\mathcal{A}))$ is **finitary compatible completion** of $\mathcal{A}$

Restriction to a duality

$$F\colon \mathbf{C_{\text{fin}}DRA} \leftrightarrows \mathbf{Stone^{+}Et}^{\text{op}} : G$$

$\mathbf{C_{\text{fin}}DRA}$: finitarily compatibly complete

$\mathbf{Stone^{+}Et}$: locally compact + 0-dimensional

Theorem (Borlido, Kudryavtseva, McLean)

There is a duality between $\mathbf{C_{\text{fin}}DRA}$ *and* $\mathbf{Stone^{+}Et}$

V: “with operators”

Additional operators

n-ary Ω on $\{-, \triangleright\}$-algebra

1. **Operator**:
   - **Normal**:
     $$\forall i \quad \Omega(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$$
   - **Additive**:
     $$\forall i \quad \Omega(\ldots, b \vee c, \ldots) = \Omega(\ldots, b, \ldots) \vee \Omega(\ldots, c, \ldots)$$

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Many natural examples: composition, domain, range, ...

Results for algebras with extra operators

$\sigma = \{\Omega, \dots\}$ symbols (for additional operations)

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- **Object:** DRA equipped with compatibility preserving operators $\Omega, \dots$
- **Morphism:** homomorphism (preserves $\{-, \triangleright\} \cup \sigma$)

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  tight, spectral, with compatibility property
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$$F : \mathbf{C}_{\mathrm{fin}}\mathbf{DRA}(\sigma) \leftrightarrows \mathbf{Stone}^{+}\mathbf{Et}(\sigma)^{\mathrm{op}} : G$$

finitary compatible completion