Diamonds and Dominoes.

Impossibility Results for Associative Modal Logics

Søren Brinck Knudstorp
ILLC and Philosophy, University of Amsterdam

September 11, 2025

TbiLLC 2025

We’ll begin with something well known (and then something that, I think, deserves to be better known)

Appetizer

Something well known:

Appetizer

Something well known: Classical Propositional Logic is decidable.

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$?

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$,

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$,

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi = (\psi \circ \varphi) \vee (\chi \circ \varphi)$,

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi = (\psi \circ \varphi) \vee (\chi \circ \varphi)$,
- $(\varphi \circ \psi) \circ \chi \leftrightarrow \varphi \circ (\psi \circ \chi)$,

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi = (\psi \circ \varphi) \vee (\chi \circ \varphi)$,
- $(\varphi \circ \psi) \circ \chi \leftrightarrow \varphi \circ (\psi \circ \chi)$,
- $$\frac{\varphi \leftrightarrow \varphi'}{\varphi \circ \psi \leftrightarrow \varphi' \circ \psi},$$

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi = (\psi \circ \varphi) \vee (\chi \circ \varphi)$,
- $(\varphi \circ \psi) \circ \chi \leftrightarrow \varphi \circ (\psi \circ \chi)$,
- $$\frac{\varphi \leftrightarrow \varphi'}{\varphi \circ \psi \leftrightarrow \varphi' \circ \psi}, \qquad \frac{\varphi \leftrightarrow \varphi'}{\psi \circ \varphi \leftrightarrow \psi \circ \varphi'}.$$

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi = (\psi \circ \varphi) \vee (\chi \circ \varphi)$,
- $(\varphi \circ \psi) \circ \chi \leftrightarrow \varphi \circ (\psi \circ \chi)$,
- $$\frac{\varphi \leftrightarrow \varphi'}{\varphi \circ \psi \leftrightarrow \varphi' \circ \psi}, \qquad \frac{\varphi \leftrightarrow \varphi'}{\psi \circ \varphi \leftrightarrow \psi \circ \varphi'}.$$

Question: Is the resulting system decidable?

Answer: It is not!

Answer: It is not! (cf. Kurucz et al. 1995)

And in fact, it is the modal logic

$$\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$$

Plan for the rest of the talk

- Setting
- Results and technique
- Related results

Plan for the rest of the talk

- Setting
- Results and technique
- Related results

Preprint available on arXiv.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$	iff	there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$: $\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ iff there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ iff there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ iff there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ iff there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ **iff** there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$\mathbb{M}, x \Vdash \varphi \circ \psi$ **iff** there exist $y, z \in X$ such that $\mathbb{M}, y \Vdash \varphi$; $\mathbb{M}, z \Vdash \psi$; and $x \in y \cdot z$.

$\cdot$ is lifted to sets $Y, Z \subseteq X$ by $Y \cdot Z := \{x \in X \mid \exists y \in Y, z \in Z : x \in y \cdot z\}$.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
    - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
    - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^+)$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic ‘hyperboolean modal logic’ and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^{+})$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^{+})$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^{+})$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^+)$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^+)$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^+)$ (raised by Bergman (2018) and Jipsen et al. (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

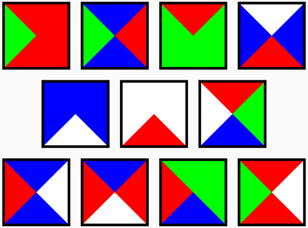


Figure 1: Wang tiles

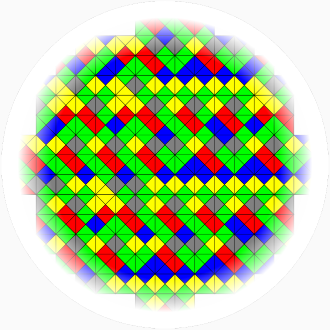


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

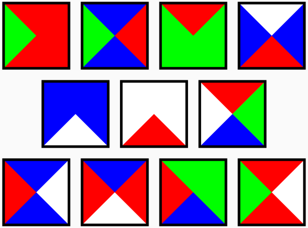


Figure 1: Wang tiles

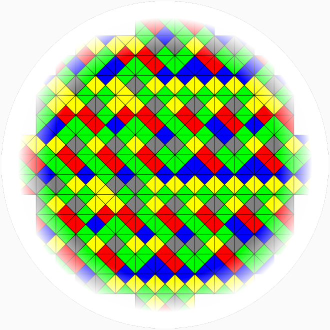


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

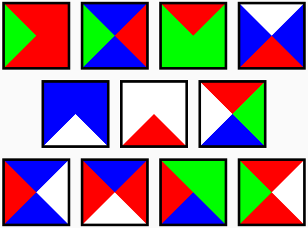


Figure 1: Wang tiles

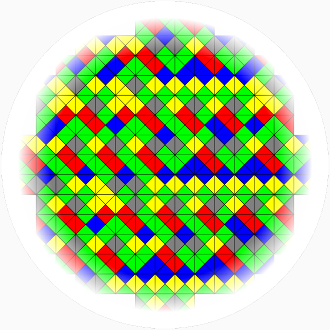


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

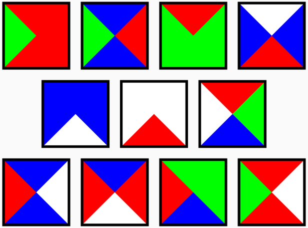


Figure 1: Wang tiles

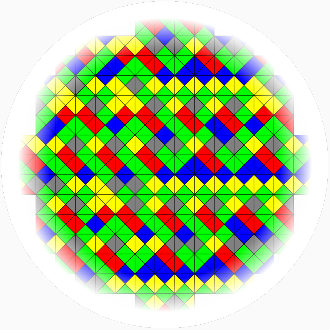


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

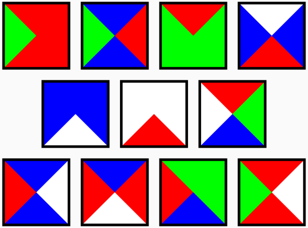


Figure 1: Wang tiles

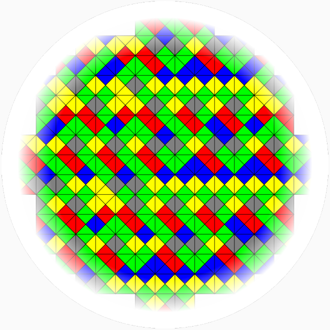


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

Main theorem

Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

Theorem

Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Main theorem

Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

$\phi_{\mathcal{W}}$ refuted by
an associative frame
(or associative BAO)

$\Nearrow$ $\Searrow$

$\phi_{\mathcal{W}}$ refuted by $(\mathcal{P}(\mathbb{N}), \cup)$
(or its complex algebra)

$\Longleftarrow$

$\mathcal{W}$ tiles
$\mathbb{N} \times \mathbb{N}$

Theorem

Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Main theorem

Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

$\phi_{\mathcal{W}}$ refuted by an associative frame (or associative BAO)

$\Nearrow$ $\quad$ $\Searrow$

$\phi_{\mathcal{W}}$ refuted by $(\mathcal{P}(\mathbb{N}), \cup)$ (or its complex algebra) $\quad \Longleftarrow \quad$ $\mathcal{W}$ tiles $\mathbb{N} \times \mathbb{N}$

Theorem

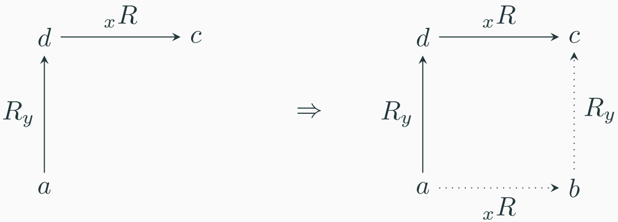
Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \text{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Associativity and Tiling 1

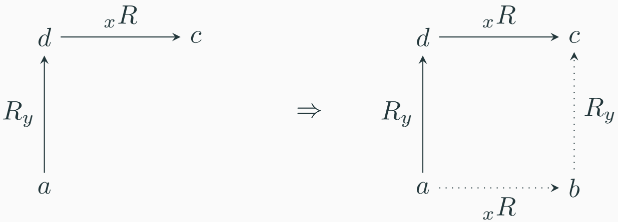
[2] ${}_zR := \{(a,b) \in X^2 \mid Razb\}$ and $R_z := \{(a,b) \in X^2 \mid Rabz\}$

Associativity and Tiling 1



[2] ${}_zR := \{(a,b) \in X^2 \mid Razb\}$ and $R_z := \{(a,b) \in X^2 \mid Rabz\}$

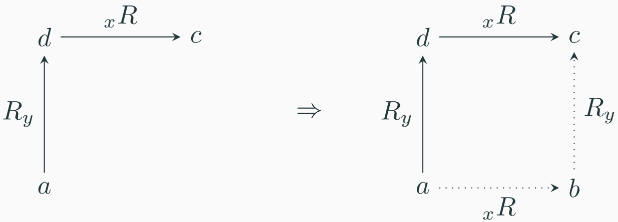
Associativity and Tiling 1



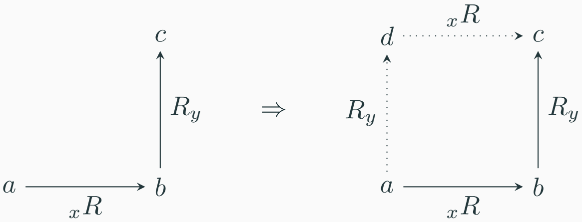
From ($Rady$ and $Rdxc$) infer $\exists b \in X$ ($Raxb$ and $Rbcy$).

[2] ${}_zR := \{(a,b) \in X^2 \mid Razb\}$ and $R_z := \{(a,b) \in X^2 \mid Rabz\}$

Associativity and Tiling 1



From ($Rady$ and $Rdxc$) infer $\exists b \in X$ ($Raxb$ and $Rbcy$).



From ($Raxb$ and $Rbcy$) infer $\exists d \in X$ ($Rady$ and $Rdxc$). [2]

[2] ${}_zR := \{(a,b) \in X^2 \mid Razb\}$ and $R_z := \{(a,b) \in X^2 \mid Rabz\}$

Associativity and Tiling 2

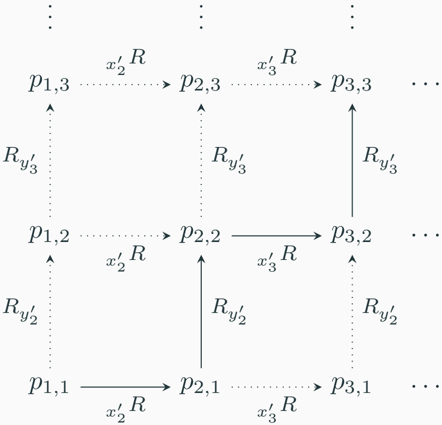


Figure 3: Generating $\mathbb{N}^2$ from the staircase: $p_{1,1}, p_{2,1}, p_{2,2}, p_{3,2}, p_{3,3}, \ldots$.

Consequences

Theorem

Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Recall the above. From this, we get:

Theorem

$\mathrm{Var}(\mathrm{BA}^+)$ is **undecidable**.

Hyperboolean modal logic is **undecidable**.

Theorem

$\mathrm{Var}(\mathrm{SL}^+)$ is **undecidable**.

The modal (information) logic of semilattices is **undecidable**.

Proof. Semilattices are associative and $(\mathcal{P}(\mathbb{N}), \cup)$ is, in particular, a semilattice.

Consequences

Theorem

Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Recall the above. From this, we get:

Theorem

$\mathrm{Var}(\mathrm{BA}^+)$ is **undecidable**.

Hyperboolean modal logic is **undecidable**.

Theorem

$\mathrm{Var}(\mathrm{SL}^+)$ is **undecidable**.

The modal (information) logic of semilattices is **undecidable**.

Proof. Semilattices are associative and $(\mathcal{P}(\mathbb{N}), \cup)$ is, in particular, a semilattice.

Consequences

Theorem

Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \mathrm{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Recall the above. From this, we get:

Theorem

$\mathrm{Var}(\mathrm{BA}^+)$ is **undecidable**.

Hyperboolean modal logic is **undecidable**.

Theorem

$\mathrm{Var}(\mathrm{SL}^+)$ is **undecidable**.

The modal (information) logic of semilattices is **undecidable**.

Proof. Semilattices are associative and $(\mathcal{P}(\mathbb{N}), \cup)$ is, in particular, a semilattice.

Other Consequences

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen et al. 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen et al. 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

New undecidability proofs for:

- $\mathbf{AK}_2 = \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ [Kurucz et al. 1993, 1995]

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen et al. 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

New undecidability proofs for:

- $\mathbf{AK}_2 = \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ [Kurucz et al. 1993, 1995]
- The classes of algebras isomorphic to (commutative) algebras of binary relations closed under composition, intersection ($\cap$), union ($\cup$), and complementation (^{c}) [Hirsch et al. 2021, Cor. 11.3]

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen et al. 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

New undecidability proofs for:

- $\mathbf{AK}_2 = \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ [Kurucz et al. 1993, 1995]
- The classes of algebras isomorphic to (commutative) algebras of binary relations closed under composition, intersection ($\cap$), union ($\cup$), and complementation (^{c}) [Hirsch et al. 2021, Cor. 11.3]
- Boolean Bunched Implication logic (BBI) [Brotherston and Kanovich 2010; Kurucz et al. 1995; Larchey-Wendling and Galmiche 2010]

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen et al. 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

New undecidability proofs for:

- $\mathbf{AK}_2 = \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ [Kurucz et al. 1993, 1995]
- The classes of algebras isomorphic to (commutative) algebras of binary relations closed under composition, intersection (∩), union (∪), and complementation (^{c}) [Hirsch et al. 2021, Cor. 11.3]
- Boolean Bunched Implication logic (BBI) [Brotherston and Kanovich 2010; Kurucz et al. 1995; Larchey-Wendling and Galmiche 2010]

Lastly: There is no translation from modal information logic into truthmaker semantics (question raised by Benthem 2017, 2024)

Related results

Thank you!

References I

Benthem, J. v. (2017). Truth Maker Semantics and Modal Information Logic. Manuscript, Institute for Logic, Language and Computation, University of Amsterdam. URL: *https://eprints.illc.uva.nl/id/eprint/1590/* (cit. on pp. 49–54).

— (2024). “Relational Patterns, Partiality, and Set Lifting in Modal Semantics”. In: Saul Kripke on Modal Logic. Ed. by Y. Weiss and R. Birman. Vol. 30. Outstanding Contributions to Logic. Springer, Cham. DOI: *10.1007/978-3-031-57635-5_5*. URL: *https://doi.org/10.1007/978-3-031-57635-5_5* (cit. on pp. 49–54).

Berger, R. (1966). The undecidability of the domino problem. English. Vol. 66. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS). DOI: *10.1090/memo/0066* (cit. on pp. 33–37).

References II

Bergman, C. (2018). "Introducing Boolean Semilattices". In: Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Scie Ed. by J. Czelakowski. Springer, pp. 103–130 (cit. on pp. 26–32, 49–54).

Brotherston, J. and M. Kanovich (2010). "Undecidability of Propositional Separation Logic and Its Neighbours". In: Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science pp. 130–139. DOI: *10.1109/LICS.2010.24*. URL: *http://dx.doi.org/10.1109/LICS.2010.24* (cit. on pp. 49–54).

Engström, F. and O. L. Olsson (2023). The propositional logic of teams. Available at *https://arxiv.org/abs/2303.14022*. arXiv: *2303.14022v3*.

Galatos, N. et al. (in preparation). Bunched Implication Logic is Undecidable.

References III

Goranko, V. and D. Vakarelov (1999). **"Hyperboolean Algebras and Hyperboolean Modal Logic".** In: Journal of Applied Non-Classical Logics 9.2-3, pp. 345–368. DOI: *10.1080/11663081.1999.10510971* (cit. on pp. 26–32).

Hirsch, R., I. Hodkinson, and M. Jackson (2021). **"Undecidability of Algebras of Binary Relations".** In: Hajnal Andréka and István Németi on Unity of Science: From Computing to Rel Ed. by J. Madarász and G. Székely. Cham: Springer International Publishing, pp. 267–287 (cit. on pp. 49–54).

Jipsen, P. (2004). **"A Note on Complex Algebras of Semigroups".** In: Relational and Kleene-Algebraic Methods in Computer Science. Ed. by R. Berghammer, B. Möller, and G. Struth. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 171–177.

References IV

Jipsen, P., M. Eyad Kurd-Misto, and J. Wimberley (2021). **"On the Representation of Boolean Magmas and Boolean Semilattices".** In: Hajnal Andréka and István Németi on Unity of Science: From Computing to Rel Ed. by J. Madarász and G. Székely. Cham: Springer International Publishing, pp. 289–312. DOI: *10.1007/978-3-030-64187-0_12* (cit. on pp. 26–32, 49–54).

Jipsen, P. and S. B. Knudstorp (in preparation). The variety of Heyting Semigroups is Decidable.

Knudstorp, S. B. (2024). **"Relevant S is Undecidable".** In: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Sc LICS '24. Tallinn, Estonia: Association for Computing Machinery. DOI: *10.1145/3661814.3662128*. URL: *https://doi.org/10.1145/3661814.3662128*.

References V

Knudstorp, S. B. (2023a). **“Logics of Truthmaker Semantics: Comparison, Compactness and Decidability”.** In: Synthese (cit. on pp. 26–32).

— (2023b). **“Modal Information Logics: Axiomatizations and Decidability”.** In: Journal of Philosophical Logic.

Kurucz, ?. et al. (1993). **“Undecidable varieties of semilattice-ordered semigroups, of Boolean algebras with operators, and logics extending Lambek calculus”.** In: Logic Journal of the IGPL 1.1, pp. 91–98. DOI: *10.1093/jigpal/1.1.91*. URL: *https://doi.org/10.1093/jigpal/1.1.91* (cit. on pp. 49–54).

References VI

- Kurucz, ?. et al. (1995). **"Decidable and Undecidable Logics with a Binary Modality".** In: Journal of Logic, Language, and Information 4.3, pp. 191–206. URL: *http://www.jstor.org/stable/40180071* (cit. on pp. 15 sq., 49–54).

- Larchey-Wendling, D. and D. Galmiche (2010). **"The Undecidability of Boolean BI through Phase Semantics".** In: Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science pp. 140–149. DOI: *10.1109/LICS.2010.18*. URL: *http://dx.doi.org/10.1109/LICS.2010.18* (cit. on pp. 49–54).

- Urquhart, A. (1972). **"Semantics for relevant logics".** In: Journal of Symbolic Logic 37, pp. 159 –169.

- — (1984). **"The undecidability of entailment and relevant implication".** In: Journal of Symbolic Logic 49, pp. 1059 –1073.

References VII

Wang, H. (1963). **"Dominoes and the $\forall\exists\forall$ case of the decision problem".** In: Mathematical Theory of Automata, pp. 23–55 (cit. on pp. 33–37).

Wang, X. and Y. Wang (2025). **"Modal logics over lattices".** In: Annals of Pure and Applied Logic 176.4, p. 103553. DOI: *https://doi.org/10.1016/j.apal.2025.103553*. URL: *https://www.sciencedirect.com/science/article/pii/S0168007225000028* (cit. on pp. 49–54).

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \mathbin{\mathbb{V}} \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash {\sim}\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1,$
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi,$
$s \vDash \varphi \vee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi,$
$s \vDash \sim\varphi$	iff	$s \nvDash \varphi,$
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi;$ $s'' \vDash \psi;$ and $s = s' \cup s''$.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1,$
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi,$
$s \vDash \varphi \vee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi,$
$s \vDash \neg\varphi$	iff	$s \nvDash \varphi,$
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi;$ $s'' \vDash \psi;$ and $s = s' \cup s''.$

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \vee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash \neg\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \circ \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$;

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{ and } \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{ or } \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1,$
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi,$
$s \vDash \varphi \vee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi,$
$s \vDash \neg\varphi$	iff	$s \nvDash \varphi,$
$s \vDash \varphi \circ \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{ and } \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{ or } \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \vee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash \neg\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \circ \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \text{ and } s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \text{ or } s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.