Topological dualities for lattices

report on joint work with Elena Pozzan

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Preamble

In logics about classical truth, whether Boolean or intuitionistic, conjunction and disjunction provides a distributive lattice structure

However, in logics where a 'fusion' operation, rather than conjunction, is residuated (satisfies MP and deduction theorem), the underlying lattice of the logic need not be distributive

Topological duality theories, allowing set theoretic models of distributive-based logics, serve as very useful tools.

This talk is about such dualities for lattices in general. In particular, we will see how they piggyback on the ones for distributive lattices

Outline

References

- Priestley spaces, Priestley duality, and canonical extension
- The filter and the ideal space of a lattice
- Discrete duality for adjunctions on downset lattices
- The free distributive lattice adjunction over a lattice
- Faithful subpolarities
- Main results linking various dualities for lattices

A few references

- Hofmann, Mislove, and Stralka, The Pontryagin Duality of Compact 0-Dimensional Semilattices and Its Applications. Lecture Notes in Mathematics **396**, Springer 1974.
- Urquhart, A topological representation theory for lattices. Algebra Universalis **8**, pages 45-58, 1978
- Hartung, A topological representation of lattices. Algebra Universalis **29**, pages 273-299, 1992
- Gehrke and van Gool, Distributive envelopes and topological duality for lattices via canonical extensions. Order **31**, pages 435-461, 2014

There are a number of surveys on the many versions of topological dualities for lattices: eg. Andrew Craig 2022, Wesley Holiday 2022, and most recently Bezhanishvili-Carai-Morandi in preprint

The different types of dualities for lattices

The different versions are all reformulations one of

1. Hofmann-Mislove-Stralka (HMS)
2. Urquhart (of which Hartung is a version)
3. Gehrke-van Gool (GvG)

Here we give a unified view of these three types based on topological ideas, more specifically, using Priestley topologies on filter and ideal spaces

An approach to this topic via the theory of canonical extensions is treated in a book on canonical extensions, which Wesley Fussner and I are currently finishing. A preliminary draft is available at `https://hal.science/hal-05231128v1`

Priestley spaces

A Priestley space is a triple $(X, \pi, \leqslant)$ so that

- (X, π) is a compact topological space
- $(X, \pi, \leqslant)$ is Totally Order Disconnected (TOD). That is, the complementary (clopen upset-clopen downset) pairs separate the points of X

PS is the category of Priestley spaces with order preserving continuous maps and DL is the category of bounded distributive lattices with bounded lattice homomorphisms

NB! $(X, \pi, \leqslant)$ is a Priestley space if and only if $(X, \pi, \geqslant)$ is one

This means there are two DLs naturally associated with a Priestley space $(X, \pi, \leqslant)$, namely

$$\mathrm{ClopDown}(\mathrm{X}, \pi, \leqslant) \textit{ and } \mathrm{ClopUp}(\mathrm{X}, \pi, \leqslant)$$

In defining the functor $\mathcal{D}\colon \mathrm{PS} \to \mathrm{DL}$, we choose the former

The Priestley functor

Let

$$\mathcal{P}\colon\ \mathrm{DL} \to \mathrm{PS}$$

be the functor that sends a DL, D, to

$$\mathcal{P}(D) = (\mathrm{PrFilt}(D), <\hat{\mathcal{B}}>, \supseteq) \cong (\mathrm{PrIdl}(D), <\check{\mathcal{B}}>, \subseteq)$$

where $\mathrm{PrFilt}(D)$ and $\mathrm{PrIdl}(D)$ are the sets of prime filters and prime ideals of D, respectively, and $\hat{\mathcal{B}}$ and $\check{\mathcal{B}}$ are the following subbases of the topologies

$$\hat{\mathcal{B}} = \{\hat{a}, (\hat{a})^c \mid a \in D\} \quad \text{and} \quad \check{\mathcal{B}} = \{\check{a}, (\check{a})^c \mid a \in D\}$$

where

$$\hat{a} = \{F \in \mathrm{PrFilt}(D) \mid a \in F\} \quad \text{and} \quad \check{a} = \{I \in \mathrm{PrIdl}(D) \mid a \notin I\}$$

Both functors, $\mathcal{D}$ and $\mathcal{P}$, are defined on morphisms by restricting and co-restricting inverse image appropriately

Priestley duality

The contravariant functors

$$\mathcal{P}\colon \mathrm{DL} \leftrightarrows \mathrm{PS}\colon \mathcal{D}$$

provide a dual equivalence of categories.

The Stone map

$\hat{(\_)}\colon D \to \mathrm{Down}(\mathrm{PrFilt}(D), \supseteq)$ or $\check{(\_)}\colon D \to \mathrm{Down}(\mathrm{PrIdl}(D), \subseteq)$

is the canonical extension of D.

Choosing downsets rather than upsets means that the order reduct of a Priestley space sits, right side up, inside the canonical extension of the dual distributive lattice

The filter space of a lattice[1]

We can define the Priestley topology on the collection of all filters for any lattice. For a lattice L, let $X_L = (\mathrm{Filt}(L), \hat{\pi}, \supseteq)$ be the poset of all filters of L equipped with the topology generated by the subbasis $\hat{\mathcal{B}} = \{\hat{a}, (\hat{a})^c \mid a \in L\}$, where

$$\hat{a} = \{x \in \mathrm{Filt}(L) \mid a \in x\}$$

for $a \in L$

Proposition: Let L be a lattice. Then X_L is a Priestley space

Question: What is the distributive lattice associated to a not-necessarily-distributive lattice L in this way?

[1] All lattices are bounded

The dual lattice of the filter space of L

$$\hat{(\ )}\colon L \to \mathcal{D}(X_L)$$

is a MSL (meet semilattice) embedding, but is not join preserving unless L is a chain.

1. $\mathcal{D}(X_L)$ is generated as a join semilattice (JSL) by $\mathrm{Im}(\hat{(\ )})$
2. The image of $\hat{(\ )}$ is the set of join irreducibles of $\mathcal{D}(X_L)$

This characterizes $\mathcal{D}(X_L)$ as the free JSL over L as a poset and/or as the free DL over L as a MSL. That is

Theorem: $\mathcal{D}(X_L) \cong JSL(L, \leqslant) \cong DL(L, \wedge)$

In words: the distributive lattice dual to the filter space of L is the free join semilattice over the meet semilattice reduct of L

The ideal space of a lattice

Using order duality, we have that

$$Y_L = (\mathrm{Idl}(L), \check{\pi}, \subseteq),$$

where $\check{\pi}$ is generated by $\check{\mathcal{B}} = \{\check{a}, (\check{a})^c \mid a \in L\}$, where $\check{a} = \{I \in \mathrm{Idl}(L) \mid a \notin I\}$ is a Priestley space and

$$(\check{\ }) \colon L \to \mathcal{D}(Y_L), a \mapsto \check{a}$$

is a JSL embedding satisfying

1. $\mathcal{D}(Y_L)$ is generated as a MSL by the image of $(\check{\ })$
2. The image of $(\check{\ })$ is the set of meet irreducibles of $\mathcal{D}(Y_L)$

That is, the distributive lattice dual to the ideal space of L is the free meet semilattice over the join semilattice reduct of L

HMS duality for lattices

The (order dual) constructions of filter space and ideal space both yield topological dualities for JSL/MSL, and, by restriction, for Lat [Hofmann-Mislove-Stralka 1974], known as

HMS duality for lattices

NB!

- L sits as a subset of its dual space
- the topology of the filter/ideal space is fully determined by the order of L
- In particular, if L is finite, then the dual space is L itself and L is recovered as its own principal downsets

Discrete duality for downset lattices

Let DL^+ denote the class of lattices isomorphic to downset lattices with complete lattice homomorphisms

NB!

- A finite lattice is in DL^+ if and only if it is distributive
- A lattice is in DL^+ if and only if it is complete, completely distributive and has enough completely join irreducible elements. That is, $J^\infty(L)$ is join dense in L

The contravariant functors

$$J^\infty\colon \mathrm{DL}^+ \leftrightarrows \mathrm{Pos}\colon \mathrm{Down}$$

provide a dual equivalence of categories.

- For finite distributive lattices,

  Discrete duality = Priestley duality = Birkhoff duality

- For lattices in general, one has the heuristic equation

  Priestley duality = Canonical extension + Discrete duality

Discrete duality for adjunctions on DL^+

Theorem: Let X and Y be posets. There is a one-to-one correspondence between

1. $R \subseteq X \times Y$ with $\geqslant \circ R \circ \geqslant = R$ (order compatible polarities)
2. Adjoint pairs of maps $f \colon \mathrm{Down}(X) \leftrightarrows \mathrm{Down}(Y) \colon g$

Here:
$f_R(U) = R[U, \_] = \{y \in Y \mid \exists x \in X \ (x \in U \text{ and } xRy)\}$
$g_R(V) = (R[\_, V^c])^c = \{x \in X \mid \forall y \in Y(xRy \implies y \in V)\}$

or, for a contravariant version, we compose with complementation on Y

$$(\ )^c \colon \mathrm{Down}(Y) \leftrightarrows \mathrm{Up}(Y)$$

to get
$f'_R(U) = (R[U, \_])^c = \{y \in Y \mid \forall x \in X \ (x \in U \implies xR^c y\}$
$g'_R(V) = (R[\_, V])^c = \{x \in X \mid \forall y \in Y(y \in V \implies xR^c y\}$
known as the Galois connection associated with R^c

A polarity between the filters and ideals of a lattice

Let $X_L = (\mathrm{Filt}(L), \hat{\pi}, \supseteq)$ and $Y_L = (\mathrm{Idl}(L), \check{\pi}, \subseteq)$ and define $R \subseteq X_L \times Y_L$ by

$$x \, R \, y \iff x \cap y = \emptyset$$

Then R is order compatible and we obtain an adjunction $f_R \colon \mathrm{Down}(X_L) \leftrightarrows \mathrm{Down}(Y_L) \colon g_R$ which is Priestley compatible. That is,

1. For each $a \in L$, $f_R(\hat{a}) = \check{a}$ is clopen in Y_L
2. For each $y \in Y_L$, $R[\_, y] = \bigcap\{\hat{a} \mid a \in y\}$ is closed in Y_L.

and in the other direction, the same holds:

3. For each $a \in L$, $g_R(\check{a}) = \hat{a}$
4. For each $x \in X_L$, $R[\_, y] = \bigcap\{\check{a} \mid a \in x\}$ is closed in X_L.

<u>Theorem</u>: (X_L, Y_L, R) is the Priestley dual of the adjunction

$$f \colon JSL(L, \leqslant) \leftrightarrows MSL(L, \leqslant) \colon g$$

given by $f(a) = a = g(a)$ for $a \in L$.The fixedpoint lattice is isomorphic to L and the fixedpoint lattice of the associated discrete adjunction is isomorphic to the canonical extension of L.

Two duality results for adjunctions

Theorem A: If X and Y are Priestley spaces, and (X, Y, R) is the Priestley dual of an adjunction

$$f\colon \mathcal{D}(X) \leftrightarrows \mathcal{D}(Y)\colon g$$

then the following are equivalent:

1. $\mathcal{D}(X)$ is generated as a JSL by the image g;
2. The principal downsets of X are fixedpoints of the associated discrete adjunction.

Order dually we have that the following are equivalent:

1. $\mathcal{D}(Y)$ is generated as a MSL by the image f;
2. The complements of principal upsets of Y are fixedpoints of the associated discrete adjunction.

Theorem B: If X and Y are Priestley spaces, and $R \subseteq X \times Y$ is the Priestley dual of an adjunction and $x\, R\, y$, then

1. there is $x' \in X$ which is minimal in $R[\_, y]$ and $x' \leqslant x$;
2. there is $y' \in Y$ which is maximal in $R[x, \_]$ and $y \leqslant y'$.

Faithfull subpolarities

Let X be a poset. If $X' \subseteq X$, we get two adjunctions

$h_\star\colon \mathrm{Down}(X') \leftrightarrows \mathrm{Down}(X)\colon h$ and $h\colon \mathrm{Down}(X) \leftrightarrows \mathrm{Down}(X')\colon h^\star$

given by

1. $h(U) = U \cap X'$ for $U \in \mathrm{Down}(X)$
2. $h_\star(V) = \downarrow_X V$ for $V \in \mathrm{Down}(X')$
3. $h^\star(V) = (\uparrow_X V)^c$ for $V \in \mathrm{Down}(X')$

Let (X, Y, R) be an order compatible polarity on posets X and Y. In general, there won't be a least subset $X' \subseteq X$ so that the fixedpoint lattice of

$$\mathrm{Down}(X') \underset{h_\star}{\overset{h}{\leftrightarrows}} \mathrm{Down}(X) \underset{f_R}{\overset{g_R}{\leftrightarrows}} \mathrm{Down}(Y)$$

is isomorphic to the fixedpoint lattice of (f_R, g_R). Similarly, there may not be a least subset and $Y' \subseteq Y$ yielding the same fixedpoints

A special case

Theorem: Let (X, Y, R) be a Priestley compatible polarity. If it satisfies the first set of equivalent conditions of Theorem A, then

$$X' = \bigcup\{\min(R[\_, y] \mid y \in Y\}$$

is the least subset of X so that the lattice of fixedpoints of the adjunction

$$\mathrm{Down}(X') \underset{h_\star}{\overset{h}{\leftrightarrows}} \mathrm{Down}(X) \underset{f_R}{\overset{g_R}{\leftrightarrows}} \mathrm{Down}(Y)$$

is isomorphic to the fixedpoint lattice of (f_R, g_R). And order dually for

$$Y' = \bigcup\{\max(R[x, \_] \mid x \in X\}$$

and

$$\mathrm{Down}(X) \underset{f_R}{\overset{g_R}{\leftrightarrows}} \mathrm{Down}(Y) \underset{g}{\overset{g^\star}{\leftrightarrows}} \mathrm{Down}(Y')$$

Main results I

In general these minimal sets X' and Y' are not closed in the Priestley topologies on X_L and Y_L, respectively. Thus they are not Priestley spaces in the induced topologies.

Theorem: Let X and Y be the topological closures of X' and Y', respectively, and let R be the relation of empty intersection on $X \times Y$. Then X and Y are the least Priestley subspaces of X_L and Y_L, respectively, so that the fixedpoint lattice of the restricted adjunctions are isomorphic to L.

This yields the GvG duality for lattices. In particular,

$$\mathcal{D}(X) = \text{the least distributive lattice containing } (L, \wedge)$$

and

$$\mathcal{D}(Y) = \text{the least distributive lattice containing } (L, \vee)$$

Main results II

Possibly more surprising is the fact that (X', Y', R) also yields a topological duality for lattices.

Theorem: Consider (X', Y', R^c) where X' is endowed with the Stone topology generated by the sets $(\hat{a})^c$ for $a \in L$ and Y' is endowed with the Stone topology generated by the sets $(\check{a})^c$ for $a \in L$. Then one can characterize the ensuing topological polarities and one recovers L as the triply closed sets ($U \subseteq X'$, which is topologically closed, $f'_R(U)$ is topologically closed, and U is Galois closed, that is, it is closed/fixed by the Galois connection (f'_R, g'_R).

This is the duality for lattices of [Urquhart 1978] in the [Hartung 1987] formulation

Summary

Given a not-necessarily-distributive lattice L, there are three different choices of topological dual space for L

(HML) X_L is the filter space of L equipped with the Priestley/Stone topology. L sits inside its dual space, the topology is given by the order. Order dually, one may use Y_L

(GvG) R_L, the relation of empty intersection between filters and ideals, is Priestley compatible. There is a least faithful Priestley compatible subpolarity (X, Y, R), i.e. with the same fixed points

(UH) Dropping the topology, the discrete polarity (X_L, Y_L, R_L) has a least faithful subpolarity (X', Y', R'). If we use the right Stone topologies on these, we also obtain a duality

When L is distributive, $X' = Y' = X = Y$ is the Stone/Priestley dual space of L and $R^c = \leqslant$, but (HML) does not agree with Priestley/Stone duality and gives a much bigger space

The spaces in the (UH) duality are not in general sober and their sobrifications are not in general spectral