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On Local Finiteness of Modal K4 Algebras

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1 Introduction

An algebra is said to be locally finite if every finitely generated subalgebra is finite. Local finiteness is a desirable property for algebras to have, as it implies the finite model property for its corresponding equational theory. The logic of locally finite modal algebras are hence guaranteed to have the finite model property and be Kripke complete, so understanding when a modal algebra is locally finite is a natural problem. For background on universal algebra and locally finite algebras, see Burris and Sankappanavar [2].

A variety is said to be locally finite if all of the algebras in it are locally finite. For varieties of modal $K4$ algebras (modal algebras validating the modal formula $\Diamond\Diamond p \to \Diamond p$), local finiteness is fairly well-studied, and is equivalent to local tabularity of its corresponding normal modal logic. A classic result by Segerberg [5] and Maksimova [4] shows that locally tabular extensions of $K4$ are precisely those which contain a formula of finite height. However, locally finite algebras need not have locally tabular logics, and in general, conditions on local finiteness of arbitrary modal algebras are not known, even for modal $K4$ algebras. In this extended abstract, we provide a few (currently) unpublished results regarding local finiteness of modal algebras, including a sufficient condition on local finiteness of modal $K4$ algebras, a classification of transitive Kripke frames whose dual complex algebras are locally finite, and some finite model property results. These findings were initially developed in the course of investigating when modal $K4$ algebras are locally finite, and as a byproduct, resolve some open questions posed by Shapirovsky [6] concerning tunable frames. We will conclude with answers to these questions.

We provide some preliminary notation. Let X be a set, $R \subseteq X \times X$ a relation. Let $\Delta_X = \{(x,x) \mid x \in X\}$ denote the diagonal of X. The relation $R \cup \Delta_X$ is the reflexive closure of R. Throughout, a modal algebra $(V, \Diamond)$ denotes a Boolean Algebra V together with a normal modal operator $\Diamond$; the Boolean operations and $\Box$ are implicitly understood. A general frame is a pair (F,V) where $F = (X,R)$ denotes a Kripke frame, and V is a collection of subsets of X such that $(V, \Diamond_R)$ is a modal algebra with $\Diamond_R A = R^{-1}[A]$. For a Kripke frame $F = (X,R)$, we similarly let the dual (complex) algebra of F be $A(F) = (\mathscr{P}(X), \Diamond_R)$. This duality is due to Jónsson and Tarski, who provided the explicit connection between modal algebras and general frames; in particular, that every modal algebra is embeddable in the complex algebra of a Kripke frame (for details and history, see, e.g., [1]). Under this correspondence, a frame and its dual modal algebra validate exactly the same modal formulas.

Rather than working with modal algebras, it is often far more tractable to work with their dual general frame. The key tool here is a property of general frames known as *tunability*. Let X be a set, R a relation on X. A partition P of X is said to be *R-tuned* if for all $A, B \in P$,

$$\exists a \in A\ \exists b \in B(aRb) \implies \forall a \in A\ \exists b \in B\ (aRb).$$

If the underlying relation is understood, we simply say that P is tuned. Accordingly, a general frame (X,R,V) is tunable if every finite partition $P \subseteq V$ of X has a finite tuned refinement $Q \subseteq V$.

A general frame is tunable iff its dual modal algebra is locally finite (see, e.g., [7], corollary 4.13 for details).

Definition 1. We say that a modal algebra $(V,\Diamond)$ contains an infinite descending $\Diamond$-tower if there exist countable nonempty, pairwise disjoint elements $T_1, T_2, T_3, \cdots \in V$ satisfying the following properties for all $i \in \mathbb{Z}_{>0}$:

(i) $T_{i+1} \subseteq \Diamond T_i$

(ii) $T_i \cap \Diamond T_{i+1} = \varnothing$

$\{T_1, T_2, \dots\}$ is accordingly referred to as an infinite descending $\Diamond$-tower. The 'tower' aspect of this construction arises from its translation into a relational setting (i.e., when considered in any general frame dual to V). Informally, T_1 forms the 'top' of the tower, with T_2 lying entirely underneath T_1, T_3 lying entirely underneath T_2, and so on. The following theorem provides a sufficient condition for local finiteness of $K4$ modal algebras.

Theorem 2. *Let $(V,\Diamond)$ be a modal $K4$ algebra. If $(V,\Diamond)$ does not contain an infinite descending $\Diamond$-tower, then it is locally finite.*

Roughly, this theorem is proven by working in a general frame dual to the modal algebra $(V,\Diamond)$, defining a procedure for refining finite partitions of the general frame which terminates only when a finite tuned refinement of the partition has been constructed. If this procedure does not terminate, it constructs an infinite descending $\Diamond$-tower, and thus, under the assumption that the algebra contains no such tower, it is always forced to terminate. Since this holds for any finite partition, the general frame is tunable and $(V,\Diamond)$ is locally finite.

The converse of this theorem is not true in general, but is true when $(V,\Diamond)$ is the complex algebra of some Kripke frame:

Proposition 3. *A complex modal $K4$ algebra is locally finite iff it does not contain an infinite descending $\Diamond$-tower.*

Recall that the *disjoint union* of a family of frames $(F_i)_{i\in I}$ where $F_i = (X_i, R_i)$ is the frame $\bigsqcup_{i\in I} F_i = (\bigsqcup_{i\in I} X_i\ , R_I)$, such that $\bigsqcup_{i\in I} X_i = \bigcup_{i\in I}(\{i\} \times X_i)$, and

$$(i,x)R_I(j,y) \quad \text{iff} \quad (i = j \ \text{ and } \ xR_iy).$$

Let D denote the disjoint union of all finite ordinals viewed as reflexive linear orders under their usual order $\leq$; that is, $D = \bigsqcup_{\alpha\in\omega}(\alpha, \leq_\alpha)$. A transitive frame (X,R) is said to be D-free if D is not a substructure of $(X, R \cup \Delta_X)$, i.e., is not a substructure of the reflexive closure of (X,R). Recall that a transitive frame is *well-founded* if it admits no infinite descending chains. We obtain the following classification:

Theorem 4. *Let F be a transitive frame. Then $A(F)$ is locally finite if and only if $A(F)$ admits an infinite descending $\Diamond$-tower if and only if F is well-founded and D-free.*

It is worth noting the relationship between local tabularity of $K4$ logics and locally finite modal $K4$ algebras. Local tabularity of a $K4$ logic is equivalent to the logic containing a formula of bounded height,

and hence all the Kripke frames it is valid over are D-free, well-founded, and Noetherian (converse well-founded). As the above theorem shows, local finiteness of an arbitrary complex $K4$ algebra is strictly more general, permitting infinite ascending chains while still forbidding infinite descending chains and D.

We obtain the following finite model property result from this theorem:

Corollary 5. *The logic of any class of well-founded transitive frames with no infinite antichains has the finite model property. In particular, the logic of any class of well-quasi-orderings (well-founded quasi-orderings with no infinite antichains) has the finite model property.*

These results can assist in answering questions about tunability of Kripke frames, hence providing more information about local finiteness of algebras and the finite model property of various logics. In particular, several open questions were raised by Shapirovsky in [6] regarding tunable Kripke frames, which can now be answered via the above theorem. One question was whether the direct product of two tunable frames is a tunable frame in general. In fact, this is not the case. Consider the frames (A,Δ_A) and $(\omega,\leq)$, where A is countable and $(\omega,\leq)$ is the standard ordering on the natural numbers. Both are tunable frames, as they are well-founded and D-free. However, their direct product is isomorphic to the disjoint union of countably many copies of $(\omega,\leq)$, which is not D-free, and hence not tunable.

It was also asked whether the algebras of the direct product of any finite family of ordinals are locally finite. A standard generalization of Dickson's Lemma [3] states that the finite direct product of well-quasi-orderings is a well-quasi-ordering. Hence, by the above theorem, we can provide a positive answer to this question:

Proposition 6. *For a finite family $(\alpha_i)_{i\in I}$ of ordinals, the complex algebras of the direct products $\prod_{i\in I}(\alpha_i,\leq)$ and $\prod_{i\in I}(\alpha_i,<)$ are both locally finite.*

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Fischer-Servi Logic Does Not Have Interpolation

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We prove that the Fischer-Servi logic IK does not have the (Craig) interpolation property. This is obtained by showing that the corresponding class of modal Heyting algebras lacks the amalgamation property. We also generalize this result to some extensions of the Fischer-Servi logic such as IT, IK4, IS4, and IGL.

The Craig interpolation property (CIP) is an important property of logical systems. For propositional logics, interpolation is closely connected to the notion of (super)amalgamation of the corresponding algebraic models of a given logic (see e.g [5, 10]). Alongside superintuitionistic and classical modal logics, intuitionistic modal logics play an important role. One of the first, and still (arguably) most influential, systems of intuitionistic modal logic was introduced by Fischer-Servi [4] (called today IK). This logic and some of its extensions, such as MIPC, have been studied extensively [12, 13, 2].

Interpolation in intuitionistic modal logic has recently attracted renewed attention, with several results known about logics like iK and its variants [6] [9, Theorem 34], such as iGL[7], as well as CK and some of its variants [8]. In this paper[1], we continue this line of work, showing that the Fischer-Servi logic and some of its close neighbours, do not have the Craig interpolation property. We prove this by demonstrating that the corresponding variety of modal Heyting algebras does not have the amalgamation property. This, in turn, is established using the duality between modal Heyting algebras and relational Esakia spaces (originally obtained by Palmigiano [11]). We also show that, with slight modifications, this example can be extended to some other intuitionistic modal logics of interest: the logics IT, IK4 and IS4, introduced by Simpson [12], and the logic IGL introduced in [3].

To show this, we recall the following algebraic semantics for Fischer-Servi logic:

Definition 1. A tuple $(H, \Box, \Diamond)$ is called an **FS**-algebra if H is a Heyting algebra and $\Box, \Diamond : H \to H$ are maps satisfying the following axioms for each $a, b \in H$:

1. $\Box 1 = 1$ and $\Diamond 0 = 0$;
2. $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$;
3. $\Diamond(a \to b) \leq \Box a \to \Diamond b$;
4. $\Diamond a \to \Box b \leq \Box(a \to b)$.

We denote by $\mathbf{FS}_{\text{alg}}$ the category of **FS**-algebras and $\Box$ and $\Diamond$-preserving Heyting homomorphisms.

Definition 2. Let $F = (X, \leq, R)$ be a tuple where $(X, \leq)$ is a poset and R a binary relation. We say that F is an **FS**-*frame* provided the following conditions are satisfied:

- **F1**: $x' \geq xRy \implies \exists y'.x'Ry' \geq y$;

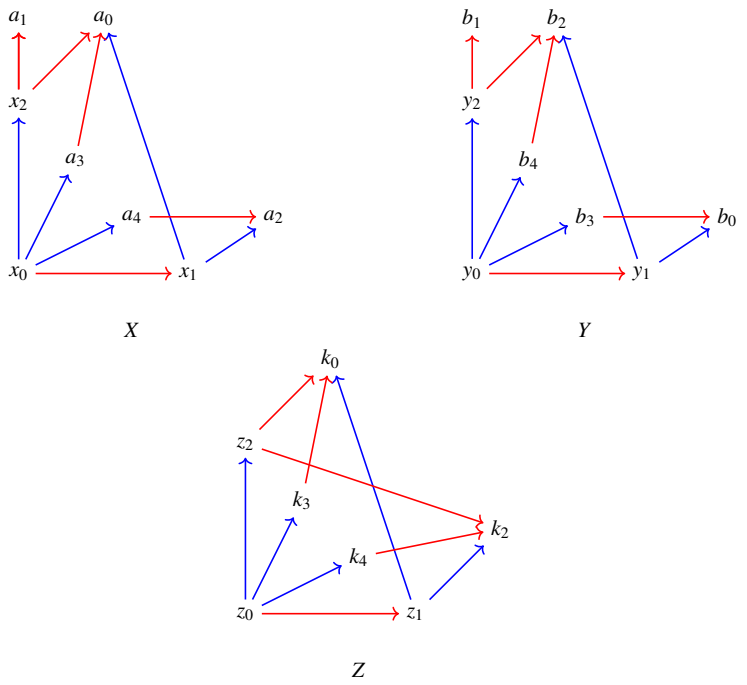

Figure 1: The co-V formation (Z,X,Y)

- **F2**: $xRy \leq y' \implies \exists x'.x \leq x'Ry'$.

Definition 3. A tuple $(X,\leq,R,\tau)$ is called an **FS**-space[2] if $(X,\leq,\tau)$ is an Esakia space, and also[3]: (1) $R[x] = R[\uparrow x] \cap \downarrow R[x]$; (2) For each $x \in X$, $R[x]$ is closed, and $R[\uparrow x]$ is a closed upset; (3) Whenever U is a clopen upset, $\Diamond_R U$ is a clopen upset, and $\Box_{\leq \circ R} U$ is a clopen upset.

Definition 4. Let $f\colon X \to Y$ be a map between **FS**-spaces. We say that f is an **FS**-morphism if (1) f is a continuous $\leq$-p-morphism; (2) if xRy, then $f(x)Rf(y)$; (3) if $f(x)Rz$, then $z \leq f(x')$ for some xRx'; (4) if $f(x) \leq m$ and mRz, then there are some $x \leq x'$ and $x'Rx''$ such that $f(x'') \leq z$.

We denote by $\mathbf{FS}_{\mathrm{sp}}$ the category of **FS**-spaces and **FS**-morphisms.

It was shown by Palmigiano [11] that the categories $\mathbf{FS}_{\mathrm{alg}}$ and $\mathbf{FS}_{\mathrm{sp}}$ are dually equivalent. Through the usual correspondence of interpolation and amalgamation properties, together with duality, we obtain:

Corollary 5. *Let L be an extension of the Fischer-Servi logic. Assume that L has the Craig interpolation property. Then whenever (X,Y_1,Y_2,f,g) is a tuple of $\mathbf{FS}_{\mathrm{L}}$-spaces such that $f\colon Y_1 \to X$ and $g\colon Y_2 \to X$ are surjective* **FS***-morphisms, then there is some $\mathbf{FS}_{\mathrm{L}}$-space W and surjective* **FS***-morphisms $p_1\colon W \to Y_1$ and $p_2\colon W \to Y_2$ such that $fp_1 = gp_2$.*

We now present our counterexample to coamalgamation for the logic IK. The three frames X,Y,Z are depicted in Figure 1. Explicitly, the blue arrows define the $\leq$-relations (with reflexive arrows omitted), whilst the red arrows denote the R-relation. The maps f and g are defined as follows:

[1] The results discussed here are reported with proofs, in [1]. We refer the reader to this preprint for unexplained notation and terminology.

[2] In [11], these are called **IK**-spaces, and the associated morphisms are called **IK**-morphisms.

[3] We note for the reader familiar with **FS**-frames, that these conditions imply the usual compatibility conditions **F1** and **F2**.

1. $f(x_i) = g(y_i) = z_i$ for $i \in \{1,2\}$.
2. $f(a_0) = k_0 = g(b_1) = g(b_0)$ and $g(b_2) = k_2 = f(a_1) = f(a_2)$.
3. $f(a_3) = k_3$ and $f(a_4) = k_4$ and $g(b_4) = k_4$ and $g(b_3) = k_3$.

The tuple (X,Y,Z,f,g) is a tuple of **FS**-spaces and morphisms, such that no **FS**-space can complete the diagram[4] Therefore, we arrive at the following result:

Theorem 6. *The Fischer-Servi logic* IK*, and the systems* IT*,* IK4*,* IS4 *and* IGL *do not have the interpolation property.*

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[4] The reader can find the details of this in [1, Theorem 4.2].

Second-Order Intuitionistic Tense Logic

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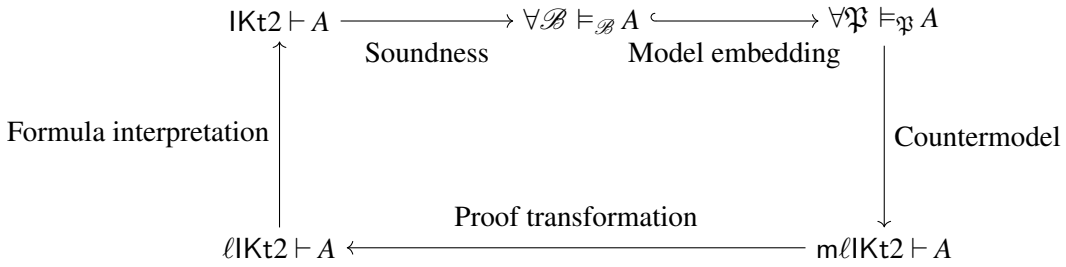
Second-order intuitionistic propositional logic (IPL2) extends propositional logic with quantification over propositions. Notably, even in the intuitionistic setting, positive connectives can be encoded with negative connectives – e.g. $A \vee B$ is equivalent to $\forall X((A \to X) \to (B \to X) \to X)$, $\bot$ is equivalent to $\forall X X$. This is contrary to intuitionistic first-order logic where the standard connectives are independent.

Intuitionistic tense logic (IKt) was first studied by Ewald [6]. In IKt, the box modality $\Box$ and the 'backwards' modality $\blacksquare$ are not De Morgan dual to their diamond counterparts $\Diamond$ and $\blacklozenge$. Whilst this was developed independently, IKt follows the 'intuitionistic' discipline introduced by Fischer Servi [7, 8], Plotkin and Stirling [11] and Simpson [13]. Other intuitionistic variants of modal logic with diamonds have been studied with no common consensus on the correct discipline, e.g. [12, 17, 3, 5, 10].

In this abstract, we define a *second-order variant of intuitionistic tense logic* (IKt2) and summarise the results given in [1]. Remarkably, with the power of second-order quantification, we can *encode* the diamonds in terms of the negative fragment of the language:

$$\Diamond A := \forall X(\Box(A \to \blacksquare X) \to X) \qquad \blacklozenge A := \forall X(\blacksquare(A \to \Box X) \to X)$$

We provide an axiomatisation, semantic foundations through birelational and predicate structures and a labelled cut-free proof system, and show their equivalence following this grand tour.



Axiomatisation. The language of IKt2 is given by the grammar: $A ::= P \in \mathsf{Prop} \mid X \in \mathsf{Var} \mid A \to A \mid \Box A \mid \blacksquare A \mid \forall X A$, where P ranges over *propositions* and X ranges over *free variables*. We assume formulas of the language are *closed*. The symbol $\bot$ and connectives $\wedge$ and $\vee$ are defined in the standard way in second-order logic, and $\Diamond$ and $\blacklozenge$ as explained above.

IKt2 is an extension of second-order propositional intuitionistic logic (IPL2) with the axioms:

$$\mathsf{D}_{\Box} : \Box(A \to B) \to \Box A \to \Box B \qquad \mathsf{D}_{\Box} : \blacksquare(A \to B) \to \blacksquare A \to \blacksquare B \qquad \mathsf{A}_{\blacksquare\Diamond} : A \to \blacksquare\Diamond A \qquad \mathsf{A}_{\Box\blacklozenge} : A \to \Box\blacklozenge A$$

and inference rules:

$$\mathsf{mp}\ \frac{A \to B \quad A}{B} \qquad \mathsf{gen}\ \frac{A[P/X]}{\forall X A}\ P \text{ fresh} \qquad \mathsf{nec}_{\Box}\ \frac{A}{\Box A} \qquad \mathsf{nec}_{\blacksquare}\ \frac{A}{\blacksquare A}$$

Unlike in first-order [9, 4] and second-order modal logics [2], the *Barcan* formula $\forall X \Box A \to \Box \forall X A$ is not required for a complete axiomatisation but is derivable.

Birelational semantics. We define *birelational semantics* similarly to the semantics of intuitionistic tense logic [6]. To account for second-order quantifiers, we must further include a domain of sets over to evaluate propositions. This restriction yields a Henkin semantics for our second-order quantifiers.

A *(second-order) birelational structure* $\mathscr{B}$ is a tuple $(W, \leq, \mathscr{W}, R_{\mathscr{B}})$ where: W is a set of worlds; $\leq$ is a pre-order on W; $\mathscr{W} \subseteq \mathscr{P}(W)$ is a class of *sets* (or *predicates*) that are upwards-closed (i.e. if $V \in \mathscr{W}$, $v \in V$ and $v \leq v'$, then $v' \in V$) and contains an interpretation $P_{\mathscr{B}} \in \mathscr{W}$ for each $P \in \mathsf{Prop}$; and $R_{\mathscr{B}} \subseteq W \times W$ is an *accessibility relation*. We furthermore require in $\mathscr{B}$ that $R_{\mathscr{B}}$ is a *bisimulation* on $\leq$, i.e.:
$\forall v, w, w' \in W\,(vR_{\mathscr{B}}w \leq w' \implies \exists v' \geq v\, v' R_{\mathscr{B}} w')$ and $\forall v, v', w \in W\,(v' \geq vR_{\mathscr{B}}w \implies \exists w' \geq w\, v' R_{\mathscr{B}} w')$

We expand the language by including each $V \in \mathscr{W}$ as a propositional symbol, setting $V_{\mathscr{B}} := V$. The judgement $v \vDash_{\mathscr{B}} A$, for $v \in W$, is defined by induction on (closed formula) A:

$v \vDash_{\mathscr{B}} P$ if $v \in P_{\mathscr{B}}$
$v \vDash_{\mathscr{B}} A \to B$ if $\forall v' \geq v\,(v' \vDash_{\mathscr{B}} A \implies v' \vDash_{\mathscr{B}} B)$
$v \vDash_{\mathscr{B}} \Box A$ if $\forall v' \geq v \forall w' \in W\,(v' R_{\mathscr{B}} w' \implies w' \vDash_{\mathscr{B}} A)$
$v \vDash_{\mathscr{B}} \blacksquare A$ if $\forall v' \geq v \forall u' \in W\,(u' R_{\mathscr{B}} v' \implies u' \vDash_{\mathscr{B}} A)$
$v \vDash_{\mathscr{B}} \forall X A$ if $\forall v' \geq v \forall V \in \mathscr{W}\ v' \vDash_{\mathscr{B}} A[V/X]$

We write simply $\vDash_{\mathscr{B}} A$ if $w \vDash_{\mathscr{B}} A$ for every $w \in W$.

$\mathscr{B}$ is *comprehensive* if, for any closed formula C (of the expanded language), it has a set $[C] = \{w \in W \mid w \vDash_{\mathscr{B}} C\}$ in $\mathscr{W}$. A *birelational model* is a comprehensive birelational structure.

Predicate semantics. We also consider intuitionistic predicate structures. A *(second-order) predicate structure* $\mathfrak{P}$ is a tuple, $(\Omega, \leq, W, \mathscr{W}, \{P^a\}_{a \in \Omega, P \in \mathscr{W}}, \{R^a\}_{a \in \Omega})$ where: Ω is a set of *intuitionistic worlds* (or *states*); $\leq$ is a partial order on Ω; W is a set of *(modal) worlds*; $\mathscr{W} \supseteq \mathsf{Prop}$ is a set of *(modal) sets* (or *predicates*); for $P \in \mathscr{W}$ and $a \in \Omega$, there is an interpretation $P^a \subseteq W$ such that if $a \leq b$, then $P^a \subseteq P^b$; and for $a \in \Omega$, $R^a \subseteq W \times W$ is an *accessibility relation* at a, such that if $a \leq b$, then $R^a \subseteq R^b$.

We similarlly expand the language of formulas by including each $V \in \mathscr{W}$ as a propositional symbol. The judgement $a, v \vDash_{\mathfrak{P}} A$, for $a \in \Omega$ and $v \in W$, is defined by induction on A:

$a, v \vDash_{\mathfrak{P}} P$ if $v \in P^a$
$a, v \vDash_{\mathfrak{P}} A \to B$ if $\forall b \geq a\,(b, v \vDash_{\mathfrak{P}} A \implies b, v \vDash_{\mathfrak{P}} B)$
$a, v \vDash_{\mathfrak{P}} \Box A$ if $\forall b \geq a \forall w \in W\,(vR^b w \implies b, w \vDash_{\mathfrak{P}} A)$
$a, v \vDash_{\mathfrak{P}} \blacksquare A$ if $\forall b \geq a \forall u \in W\,(uR^b v \implies b, u \vDash_{\mathfrak{P}} A)$
$a, v \vDash_{\mathfrak{P}} \forall X A$ if $\forall b \geq a \forall P \in \mathscr{W}\ b, v \vDash_{\mathfrak{P}} A[P/X]$

We write simply $\vDash_{\mathfrak{P}} A$ if $a, v \vDash_{\mathfrak{P}} A$ for every $a \in \Omega, v \in W$.

$\mathfrak{P}$ is *comprehensive* if, for each formula C (of the expanded language), it has a set $[C] \in \mathscr{W}$ with $[C]^a = \{w \in W \mid a, w \vDash_{\mathfrak{P}} C\}$. A *predicate model* is a comprehensive predicate structure.

Labelled proof systems. We consider a *labelled* proof system for IKt2 adapting Simpson's calculus for IK [13] to tense modalities and second-order quantifiers [15]. Fix a set WI of *world symbols*, written u, v, w etc. A *relational atom* is an expression vRw, where $v, w \in \mathsf{WI}$. A *labelled formula* is an expression $v : A$ where $v \in \mathsf{WI}$ and A is a formula. A *labelled sequent* is an expression $\mathbf{R} \,|\, \Gamma \Rightarrow \Delta$ where $\mathbf{R}$, the *relational context*, is a set of relational atoms and Γ, Δ are multisets of labelled formulas, respectively the *antecedent* and *succedent*.

The system mℓIKt2 is given by the rules in figure 2. where every upper sequent of a right-rule must have a succedent of size at most one. The system ℓIKt2 has the same rules as mℓIKt2 with the restriction that all sequents must have succedent of size at most one.

$$\mathsf{id}\,\frac{}{\mathbf{R}\,|\,v:A \Rightarrow v:A}$$

$$\mathsf{w}_l\,\frac{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta}{\mathbf{R}\,|\,\Gamma, v:A \Rightarrow \Delta} \qquad \mathsf{w}_r\,\frac{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:A}$$

$$\mathsf{c}_l\,\frac{\mathbf{R}\,|\,\Gamma, v:A, v:A \Rightarrow \Delta}{\mathbf{R}\,|\,\Gamma, v:A \Rightarrow \Delta} \qquad \mathsf{c}_r\,\frac{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:A, v:A}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:A}$$

$$\to_l\,\frac{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:A \quad \mathbf{R}\,|\,\Gamma', v:B \Rightarrow \Delta'}{\mathbf{R}\,|\,\Gamma, \Gamma', v:A \to B \Rightarrow \Delta, \Delta'} \qquad \to_r\,\frac{\mathbf{R}\,|\,\Gamma, v:A \Rightarrow v:B}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:A \to B}$$

$$\forall_l\,\frac{\mathbf{R}\,|\,\Gamma, v:A[C/X] \Rightarrow \Delta}{\mathbf{R}\,|\,\Gamma, v:\forall XA \Rightarrow \Delta} \qquad \forall_r\,\frac{\mathbf{R}\,|\,\Gamma \Rightarrow v:A[P/X]}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:\forall XA}\ P \text{ fresh}$$

$$\Box_l\,\frac{\mathbf{R}, vRw\,|\,\Gamma, w:A \Rightarrow \Delta}{\mathbf{R}, vRw\,|\,\Gamma, v:\Box A \Rightarrow \Delta} \qquad \Box_r\,\frac{\mathbf{R}, vRw\,|\,\Gamma \Rightarrow \Delta, w:A}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:\Box A}\ w \text{ fresh}$$

$$\blacksquare_l\,\frac{\mathbf{R}, uRv\,|\,\Gamma, u:A \Rightarrow \Delta}{\mathbf{R}, uRv\,|\,\Gamma, v:\blacksquare A \Rightarrow \Delta} \qquad \blacksquare_r\,\frac{\mathbf{R}, uRv\,|\,\Gamma \Rightarrow \Delta, u:A}{\mathbf{R}\,|\,\Gamma \Rightarrow \Delta, v:\blacksquare A}\ u \text{ fresh}$$

Figure 2: Rules of mℓIKt2 and ℓIKt2. A symbol is *fresh* if it does not occur in the lower sequent.

Summary of results. To link the structures introduced above, and establish the grand tour displayed at the beginning, we prove that the following are equivalent:

1. IKt2 $\vdash A$;
2. $\vDash_{\mathscr{B}} A$ for all birelational models $\mathscr{B}$;
3. $\vDash_{\mathfrak{P}} A$ for all predicate models $\mathfrak{P}$;
4. mℓIKt2 $\vdash \cdot\,|\,\cdot \Rightarrow w:A$ for a world symbol w;
5. ℓIKt2 $\vdash \cdot\,|\,\cdot \Rightarrow w:A$ for a world symbol w.

We give a brief proof sketch; for full details, the reader is referred to [1].

1. $\Longrightarrow$ 2.: standard soundness argument by induction on the proof of IKt2 $\vdash A$.

2. $\Longrightarrow$ 3.: we can view predicate models as birelational structures by essentially taking the product $\Omega \times W$ for the set of worlds of a birelational model. By induction on A we show this embedding faithful.

3. $\Longrightarrow$ 4.: cut-free completeness of the calculus mℓIKt2 is shown by a countermodel construction. Informally, this is done by iterating two proof search phases: (i) a transfinite, invertible and non-branching phase, and (ii) a single-step, infinitely branching and non-invertible phase. The transfiniteness of proof search is due to the comprehension rule $\forall_l$ where C can be an arbitrary formula. From this transfinite tree, we can read out a predicate counter-model with non-invertible phases corresponding to intuitionistic worlds, while labels and relational atoms correspond to modal worlds and their modal relations. Finally, in order to define an appropriate domain of sets, we exploit techniques by Prawitz and Tait for completing semivaluations, à la Schütte.

4. $\Longrightarrow$ 5.: by induction on the mℓIKt2 proof of $\cdot\,|\,\cdot \Rightarrow w:A$. Due to the absence of native positive connectives, namely $\vee$, this is much easier than the first-order and propositional cases [16].

5. $\Longrightarrow$ 1.: we define a formula interpretation of labelled sequents, and we show that rules are sound in IKt2 through the interpretation similarly to [14]. This relies on the fact that $\mathbf{R}$ is a rooted (poly-)tree

whose vertices we can interpret in the modal language by tense modalities.

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On the Proof Theory of the Constructive μ-Calculus

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Modal logic presents an interesting case in the landscape of logical theory, partially because while there are interesting consequences resulting from enforcing an intuitionistic version of modal logic, there is no singular approach to achieve this. Depending on the nature of the underlying motivation, the constructed system satisfies different axioms and properties [6]. Instead of considering axioms that should be fulfilled by the intuitionistic modal logic [3], a different, proof-theoretically driven approach is to place the regular intuitionistic restriction of only one formula on the right hand side on the sequent calculus for classical modal logic [13]. This results in what is commonly called constructive modal logic, also referred to as CK.

The study of logics with least and greatest fixed points represents a significant trend in recent research. Many of these logics lie in the modal realm, with Kozen's modal μ-calculi [9] being a central part of this development. Notably, an underlying intuitionistic motivated logic has several implications for the added fixed points, for example see [4, 7].

When constructing a proof system for logics including fixed points, ill-founded proof systems have emerged as well suited to the requirements [11]. In such systems, proof trees with infinite height are allowed, but placed under an external constriction. This greatly simplifies the rules for the fixed points operators, which can now simply consist of the unfolding of said fixed point. To ensure soundness, appealing to local soundness is no longer sufficient. Instead, the external constriction most often takes the form of the global trace condition, also referred to as the progressivity condition. Hereby, the condition imposes that some feature occurs infinitely often on all infinite branches in a proof tree.

In light of this, it seems natural to examine the logic constructed by adding least and greatest fixed points to an intuitionistic modal logic [2]. The close alignment of CK with its proof theory and the prior, albeit limited, usage of its extension with fixed points lead us to the choice of CK as the basis. We will refer to this constructive μ-calculus as $\mu\mathsf{CK}$.

Little is known about $\mu\mathsf{CK}$. Prior to this work, this logic only finds mention in two papers. In a 2024 paper, Pacheco introduces game semantics for $\mu\mathsf{CK}$ and relates them to their Kripke-style semantic counterpart [12]. As far as we know, this is the only work in which this logic is the main focus. A (weak) fragment of $\mu\mathsf{CK}$ has a prominent role in the proof of completeness for the (classical) modal μ-calculus by Afshari et al [1] where the constraints on formal proofs implicit to intuitionistic logic bootstrap completeness for the classical logic.

Another motivation comes from developments towards a general cut elimination procedure for ill-founded proof system in a recent paper by Curzi and Leigh [5]. The method presented there is developed and demonstrated on $\mu\mathsf{MALL}$, the multiplicative and additive fragment of linear logic extended with

fixed points. The argument heavily uses the dualities of the connectives in μMALL and is based on an one-sided sequent calculus. To adapt this approach for other logics, many elements of the proof need to be adjusted.

Two major differences between μMALL and μCK are the inclusion of modalities and the missing interdefinability between $\Box$ and $\Diamond$, and μ and ν, which results in a two-sided sequent calculus. Therefore, μCK serves as a good case study for extending the strategy in [5] for different logics. Establishing the result of cut elimination for μCK would provide an example of how to achieve this for other intuitionistic logics, as well as for logics with modalities.

All together, our work defines μCK, the underlying constructive modal logic and the fixed points, with its Kripke-style and game semantics. After providing an ill-founded proof system for it, we adapt the argument in [5] for this new setting to show general cut elimination for μCK.

The syntax of μCK adheres to the expected structure. We follow the convention of using a capital letter X for variables and a lower case letter p for propositional symbols.

Definition 1. A *formula* is generated by the following grammar:

$$\varphi \coloneqq p \mid \bot \mid \top \mid X \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

Notable, $\Box, \Diamond, \mu$ and ν are all part of the syntax, since they cannot be defined in terms of each other. To ensure a sound definition for the fixed points, we restrict the fixed point variable to only occur positively in its associated fixed point. Since our syntax does not include negation, positivity refers to the formula occurring in the antecedent of an implication an even number of times.

The Kripke-style semantics for the CK-fragment follow the historic precedent, set in the introductory work for constructive modal logic by Wijesekera [13], with the slight adjustment of allowing for fallible worlds that satisfy $\bot$ [10]. The semantics for the fixed points are added in the usual way, as fixed points of monotone functions, which are well-defined by the Knaster-Tarski theorem. The latter, and both the introduction and the proof of equivalence with game semantics can be found in [12].

To form our ill-founded system for μCK, we adjust the finitary system from [13] by only allowing exactly one formula on the right hand side[1] and add the standard rules for fixed point unfoldings. Since they are symmetric, we present the general fixed point rules for $\sigma \in \{\mu, \nu\}$. Figure 3 shows the rules of our proof system.

To to ensure soundness, we add the global trace condition. A trace is a sequence of formulas, starting at some formula in a branch and following the ancestors of this formula upwards through the branch. A progressing derivation or proof is a derivation where for every infinite branch, there exists a trace through that branch such that along this trace, a least fixed point unfolds infinitely often on the left hand side of the sequent, or a greatest fixed points unfolds infinitely often on the right hand side.

Towards cut elimination, specifying an infinitary cut reduction procedure for ill-founded proof systems, even along the infinite branches, is straightforward. Cut elimination is then obtained as a limit of such an infinitary procedure. The challenge lies in showing that such a limit cut-free derivation is both well defined and also still progressing. To illustrate this issue, consider a proof where the trace showing progressivity along some infinite branch begins in a cut formula. After applying the cut reductions, this cut formula and its ancestors will not be included in the cut-free derivation. Since the originial trace does no longer exist, what is now witnessing the progressivity of the derivation along this infinite branch?

The cut elimination argument from [5] and adapted here is based on reducibility candidates, a concept introduced by Tait [14] and Girard [8] in the context of lambda calculus. Simply put, reducibility

[1] This adjusted sequent calculus can also be found in [6], where it is referred to as LCK

$$\frac{\Gamma, \varphi, \psi, \Gamma' \Rightarrow \delta}{\Gamma, \psi, \varphi, \Gamma' \Rightarrow \delta}\,\mathsf{ex} \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \delta}{\Gamma, \varphi \Rightarrow \delta}\,\mathsf{co} \qquad \frac{\Gamma \Rightarrow \delta}{\Gamma, \varphi \Rightarrow \delta}\,\mathsf{wk} \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Lambda, \varphi \Rightarrow \delta}{\Gamma, \Lambda \Rightarrow \delta}\,\mathsf{cut}$$

$$\frac{}{\bot \Rightarrow \delta}\,\bot \qquad \frac{}{\Rightarrow \top}\,\top \qquad \frac{}{p \Rightarrow p}\,\mathsf{id} \qquad \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\,\Box \qquad \frac{\Gamma, \varphi \Rightarrow \delta}{\Box\Gamma, \Diamond\varphi \Rightarrow \Diamond\delta}\,\Diamond$$

$$\frac{\Gamma, \varphi \Rightarrow \delta \qquad \Gamma, \psi \Rightarrow \delta}{\Gamma, \varphi \vee \psi \Rightarrow \delta}\,\vee_{\mathsf{L}} \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}\,\vee_{\mathsf{R}} \qquad \frac{\Gamma, \varphi_i \Rightarrow \delta}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \delta}\,\wedge_{\mathsf{L}}$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\,\wedge_{\mathsf{R}} \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \delta}{\Gamma, \varphi \to \psi \Rightarrow \delta}\,\to_{\mathsf{L}} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\,\to_{\mathsf{R}}$$

$$\frac{\Gamma, \varphi(\sigma X.\varphi(X)) \Rightarrow \delta}{\Gamma, \sigma X.\varphi(X) \Rightarrow \delta}\,\sigma_{\mathsf{L}} \qquad \frac{\Gamma \Rightarrow \varphi(\sigma X.\varphi(X))}{\Gamma \Rightarrow \sigma X.\varphi(X)}\,\sigma_{\mathsf{R}}$$

Figure 3: The sequent calculus of $\mu\mathsf{CK}$

candidates model sets of objects that normalize in some way. In our case, they serve as a proof-theoretic interpretation of $\mu\mathsf{CK}$ formulas, consisting of certain derivations of the given formula that reduce under this cut reduction procedure to a sound and cut-free derivation. We write $[\![\varphi]\!]$ as the reducibility candidate interpretation of the formula φ. This interpretation is defined by an induction on the structure of φ and is of a syntactic nature. As an example, we give the definition of the interpretation of $\Diamond\varphi$:

$$[\![\Diamond\varphi]\!] := \left\{ \left. \frac{\begin{array}{c} d \\ \Gamma, \psi \Rightarrow \varphi \end{array}}{\Box\Gamma, \Diamond\psi \Rightarrow \Diamond\varphi}\,\Diamond \;\right|\; d \in [\![\varphi]\!] \right\}^*$$

The interpretation of $\Diamond\varphi$ is constructed by taking derivation in the interpretation of φ and applying the $\Diamond$-rule to them. The $*$-operator closes the definition under cut reduction, meaning that every derivation that reduces to a derivation belonging in this set, also belong to the set. We can also establish the following lemma from the definition of this interpretation:

Lemma 2 (Normalization)**.** *Every derivation d in a reducibility interpretation $[\![\varphi]\!]$ can be rewritten by a cut reduction procedure into a progressing and cut-free proof.*

We aim to show that every derivation is contained in some reducibility candidate. First, we establish the correct behaviour of our proof system in regards to the interpretation with the following lemma:

Lemma 3 (Soundness)**.** *The rules of the sequent calculus of $\mu\mathsf{CK}$ are sound with respect to the interpretation based on reducibility candidates.*

We can prove this by a straightforward case distinction on the last rule of a derivation. Next, we want to show our main theorem, whose immediate corollary, together with the normalization lemma, is that every derivation reduces to one without cuts:

Theorem 4. *Every proof d with conclusion $\Rightarrow \varphi$ is in the reducibility interpretation of φ, $[\![\varphi]\!]$.*

As is common in arguments involving fixed points, the proof strategy of this theorem will hinge on ordinal approximates of fixed points. Additionally, we argue by contraposition. Assuming a derivation d does not belong to a suitable reducibility candidate and repeatedly applying the soundness lemma yields an infinite branch in d. Since d is a proof and therefore progressing, there is a trace along this branch

with the correct fixed point unfoldings. From these, using the ordinal approximations, we can construct an infinitely descending chain of ordinals, which contracts the well-foundedness of ordinals.

This proof method is non-constructive. Our cut elimination argument only states the existence of a cut reduction procedure with a cut-free limit derivation without stating precisely which. This also avoids the problem of showing the progressivity of the limit derivation.

A natural continuation of this work would be the adaptation of the second cut elimination argument in [5]. With a stricter variation of progressivity placed on derivations and some additional results, the authors extend the ideas behind the approach presented here. Essentially, this variation encodes well-behaved cut reduction in the infinitary case into the progressivity condition. Showing that both this and the global trace condition agree on every derivation results in an explicit procedure for cut elimination.

Furthermore, in [1] several results are established for a fragment of $\mu\mathsf{CK}$ essentially corresponding to the implication free fragment. Most notably, cut elimination, a normal form theorem and completeness are shown for this fragment. While in our work we lift cut elimination to the entirety of $\mu\mathsf{CK}$, a possible future endeavour consists of finding a precise formulation of the normal form theorem which can be applied to the entirety of $\mu\mathsf{CK}$.

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A Terminating Sequent Calculus for Kuznetsov-Muravitski Logic

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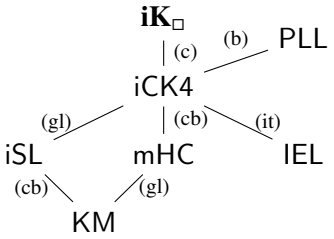
Within the framework of intuitionistic modal logics, systems that consider only the $\Box$ modality and satisfy the *normality principle* (k) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and the *coreflection principle* (c) $\varphi \to \Box\varphi$ have attracted considerable attention due to their connections with provability interpretations [13], epistemic interpretations [1, 11], as well as their applications in formal verification [5]. The following figure provides a schematic overview of these systems, their axiomatizations and their mutual relationships starting from the $\Box$-fragment of intuitionistic modal logic **iK**; see [14] for a systematic review.

(c) $\varphi \to \Box\varphi$
(cb) $\Box\varphi \to ((\psi \to \varphi) \vee \psi)$
(it) $\Box\varphi \to \neg\neg\varphi$
(gl) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$
(b) $\Box\Box\varphi \to \Box\varphi$



Adopting the normality and coreflection principles has important consequences for the usual Kripke birelational semantics of intuitionistic modal logics. In the resulting models, called *strong*, the modal accessibility relation R is constrained by the intuitionistic one ($R \subseteq \leq$); as a consequence, the persistence of forcing, which characterizes the $\leq$ relation in intuitionistic Kripke models, also holds for R.

Within this landscape, we aim at developing terminating sequent calculi that support constructive reasoning for all the logics of the above diagram, within the framework of labelled calculi for intuitionistic logic developed in [6]. In this direction, we have already treated the *minimal coreflection logic* iCK4 [8] (also known as IEL$^-$ in the epistemic setting [1]) and the Intuitionistic Strong Löb Logic iSL [9]. Here we consider the case of the *Kuznetsov–Muravitsky Logic* KM, which has been investigated for its connections with the provability logics [12]. KM is the logic obtained by extending iSL with the Cantor–Bendixson axiom $\Box\varphi \to ((\psi \to \varphi) \vee \psi)$, and is semantically characterized by the strong Kripke models $\mathscr{K} = \langle W, \leq, R, r, V\rangle$ in which the modal accessibility relation R coincides with $<$ (the strict order included in the intuitionistic relation $\leq$). The KM-consequence relation $\models_{\mathsf{KM}}$ is defined as follows: $\Gamma \models_{\mathsf{KM}} \varphi$ iff $\forall \mathscr{K}\ \forall w \in \mathscr{K}\ \ (w \Vdash \Gamma \implies w \Vdash \varphi)$.

$$\frac{}{\Gamma \overset{l}{\Rightarrow} \alpha\,;\,\psi}\ \mathrm{Ax}^{\triangleright}\ \text{if } \Gamma \triangleright \alpha \qquad \frac{}{\bot,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\bot$$

$$\frac{\alpha,\beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha\wedge\beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\wedge \qquad \frac{\Gamma \overset{l}{\Rightarrow} \alpha\,;\,\psi \qquad \Gamma \overset{l}{\Rightarrow} \beta\,;\,\psi}{\Gamma \overset{l}{\Rightarrow} \alpha\wedge\beta\,;\,\psi}\ R\wedge$$

$$\frac{\alpha,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta \qquad \beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha\vee\beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\vee \qquad \frac{\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha_k\,;\,\alpha_0\vee\alpha_1}{\Gamma \overset{\mathrm{u}}{\Rightarrow} \alpha_0\vee\alpha_1}\ R^{\mathrm{u}}_{\vee_k} \qquad \frac{\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha_k\,;\,\psi}{\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha_0\vee\alpha_1\,;\,\psi}\ R^{\mathrm{b}}_{\vee_k}$$

$$\frac{\alpha\to\beta,\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha\,;\,\delta \qquad \beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha\to\beta,\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\to \qquad \frac{\Gamma \overset{l}{\Rightarrow} \beta\,;\,\psi}{\Gamma \overset{l}{\Rightarrow} \alpha\to\beta\,;\,\psi}\ R^{\triangleright}_{\to}\ \text{if } \Gamma \triangleright \alpha$$

$$\frac{\alpha,\Gamma,\Box\Theta \overset{\mathrm{u}}{\Rightarrow} \beta \quad \alpha,\Gamma,\Theta \overset{\mathrm{u}}{\Rightarrow} \beta}{\Gamma,\Box\Theta \overset{\mathrm{u}}{\Rightarrow} \alpha\to\beta}\ R^{\mathrm{u}}_{\to}\ \text{if } \Gamma\cup\Box\Theta \not\triangleright \alpha \qquad \frac{\alpha,\Gamma,\Box\Theta \overset{\mathrm{u}}{\Rightarrow} \psi \quad \alpha,\Gamma,\Theta \overset{\mathrm{u}}{\Rightarrow} \beta}{\Gamma,\Box\Theta \overset{\mathrm{b}}{\Rightarrow} \alpha\to\beta\,;\,\psi}\ R^{\mathrm{b}}_{\to}\ \text{if } \Gamma\cup\Box\Theta \not\triangleright \alpha$$

$$\frac{\Gamma,\Theta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma,\Box\Theta \overset{\mathrm{u}}{\Rightarrow} \Box\alpha}\ R^{\Box}_{\mathrm{u}} \qquad \frac{\Box\alpha,\Gamma,\Theta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma,\Box\Theta \overset{\mathrm{b}}{\Rightarrow} \Box\alpha\,;\,\psi}\ R^{\Box}_{\mathrm{b}}\ \text{if } \Gamma\cup\Box\Theta \not\triangleright \Box\alpha$$

Figure 4: The calculus $\mathscr{E}$-KM ($l \in \{\mathrm{b},\mathrm{u}\}$, $k \in \{0,1\}$).

Our approach to designing a terminating calculus for KM relies on decorating sequents with suitable labels to control termination and on equipping the calculus with an evaluation relation $\triangleright$ between a set of formulas Γ and a formula φ guiding rule application. Namely, $\Gamma \triangleright \varphi$ (Γ *evaluates* φ) iff φ matches the following BNF: $\varphi := \gamma \mid \varphi\wedge\varphi \mid \varphi\vee\alpha \mid \alpha\vee\varphi \mid \alpha\to\varphi \mid \Box\varphi$ with $\gamma\in\Gamma$ and α any formula.

Fig. 4 displays the calculus $\mathscr{E}$-KM for KM. It acts on labelled sequents of the form $\Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$ or $\Gamma \overset{\mathrm{b}}{\Rightarrow} \delta\,;\,\psi$, where Γ is a multiset of formulas and δ,ψ are formulas. $\mathscr{E}$-KM consists of the axiom rules $\mathrm{Ax}^{\triangleright}$ and $L\bot$, left/right rules for propositional connectives, and right rules for $\Box$; left occurrences of $\Box$ are handled implicitly by the $\Box$-right rules and by $R^{\mathrm{u}}_{\to}$ and $R^{\mathrm{b}}_{\to}$. With $\Gamma \overset{l}{\Rightarrow} \delta\,;\,\psi$ where $l\in\{\mathrm{b},\mathrm{u}\}$, we denote a generic l-sequent standing for either an u- or a b-sequent; if $l=\mathrm{u}$, then ψ is omitted. E.g., $R^{\triangleright}_{\to}$ applied to a b-sequent has conclusion $\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha\to\beta\,;\,\psi$ and premise $\Gamma \overset{\mathrm{b}}{\Rightarrow} \beta\,;\,\psi$, while on an u-sequent it has conclusion $\Gamma \overset{\mathrm{u}}{\Rightarrow} \alpha\to\beta$ and premise $\Gamma \overset{\mathrm{u}}{\Rightarrow} \beta$.

$\mathscr{E}$-KM is oriented to backward proof search, where rules are applied bottom-up. If the conclusion of a rule has label b (where b stands for *blocked* and u for *unblocked*), the (bottom-up) application of left rules is blocked. There are three rules for right implication, namely $R^{\triangleright}_{\to}$, $R^{\mathrm{u}}_{\to}$ and $R^{\mathrm{b}}_{\to}$; the choice between them is settled by the evaluation relation $\triangleright$ and the label of the conclusion. We write $\vdash \mathscr{E}\text{-KM}\sigma$ to mean that the sequent σ is provable in $\mathscr{E}$-KM.

Fig. 5 displays the $\mathscr{E}$-derivation of the u-sequent $\sigma_0 = \overset{\mathrm{u}}{\Rightarrow} \alpha\wedge\beta$ where α is an instance of the Cantor–Bendixson axiom and β is an instance of the *strong Gödel Löb axiom* [14]. Sequents in the derivation are indexed by (n) and referred to as σ_n. In the applications of $R^{\mathrm{u}}_{\to}$ with conclusions σ_1 and σ_6, the two premises coincide, so only one is displayed. This example illustrates some key features of $\mathscr{E}$-KM. First, the formula $\gamma = (p\to q)\vee q$ in the premise σ_3 of $R^{\mathrm{u}}_{\vee_0}$, together with its treatment by $R^{\mathrm{b}}_{\to}$, is crucial for deriving α. Both disjuncts of γ are essential: σ_5 is an axiom because of the first disjunct, stored in the left-hand side of σ_3, whereas σ_4 is an axiom because of the second disjunct, stored in the context of σ_3 and moved to the right-hand side by $R^{\mathrm{b}}_{\to}$. A similar effect can be obtained with a multi-succedent calculus [10]; however, multiple formulas on the right-hand side increase the nondeterminism,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q,\Box p \overset{u}{\Rightarrow} (q\to p)\vee q_{(4)}}\,\mathrm{Ax}^{\triangleright} \quad \dfrac{}{q,p \overset{u}{\Rightarrow} p_{(5)}}\,\mathrm{Ax}^{\triangleright}}{\Box p \overset{b}{\Rightarrow} q\to p\,;(q\to p)\vee q_{(3)}}\,R\overset{b}{\to}}{\Box p \overset{u}{\Rightarrow} (q\to p)\vee q_{(2)}}\,R^{u}_{\vee 0}}{\overset{u}{\Rightarrow}\Box p\to((q\to p)\vee q)_{(1)}}\,R\overset{u}{\to} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p,\Box p\to p\overset{b}{\Rightarrow}\Box p\,;p_{(10)}}\,\mathrm{Ax}^{\triangleright}}{\Box p,\Box p\to p\overset{u}{\Rightarrow} p_{(9)}}\,R^{\Box}_{b} \quad \dfrac{}{\Box p,p\overset{u}{\Rightarrow} p_{(11)}}\,\mathrm{Ax}^{\triangleright}}{\Box p\to p\overset{b}{\Rightarrow}\Box p\,;p_{(8)}}\,L\to \quad \dfrac{}{p\overset{u}{\Rightarrow} p_{(12)}}\,\mathrm{Ax}^{\triangleright}}{\Box p\to p\overset{u}{\Rightarrow} p_{(7)}}\,L\to}{\overset{u}{\Rightarrow}(\Box p\to p)\to p_{(6)}}\,R\overset{u}{\to}}{\overset{u}{\Rightarrow}(\Box p\to((q\to p)\vee q))\wedge((\Box p\to p)\to p)_{(0)}}\,R\wedge$$

Figure 5: The $\mathscr{E}$-derivation $\mathscr{D}$ of $\sigma_0 = \overset{u}{\Rightarrow}(\Box p\to((q\to p)\vee q))\wedge((\Box p\to p)\to p)$.

and hence the complexity, of proof search. Second, the b label in σ_8, obtained from σ_7 by $L\to$, blocks further applications of $L\to$, which would otherwise generate an infinite branch. Third, note the role of the diagonal formula $\Box p$ in σ_9 obtained from σ_8 by $R^{\Box}_{b}$; without it, σ_9 would be $\Box p\to p\overset{u}{\Rightarrow} p$ and, after applying $L\to$, the left premise would be $\sigma'_{10} = \Box p\to p\overset{b}{\Rightarrow}\Box p\,;p$, yielding the loop $\sigma'_{10} = \sigma_8$.

The main properties of $\mathscr{E}$-KM are the following:

Theorem

(i) $\mathscr{E}$-KM *has the subformula property.*

(ii) $\mathscr{E}$-KM *is strongly terminating.*

(iii) $\vdash_{\mathscr{E}\text{-KM}} \Gamma\overset{u}{\Rightarrow}\delta$ *implies* $\Gamma\models_{\mathsf{KM}}\delta$ *and* $\vdash_{\mathscr{E}\text{-KM}} \Gamma\overset{b}{\Rightarrow}\delta\,;\psi$ *implies* $\Gamma\models_{\mathsf{KM}}\delta\vee\psi$ *(Soundness).*

(iv) $\Gamma\models_{\mathsf{KM}}\delta$ *implies* $\vdash_{\mathscr{E}\text{-KM}}\Gamma\overset{u}{\Rightarrow}\delta$ *(Completeness).*

Strong termination means that backward proof search in $\mathscr{E}$-KM always terminates, regardless of the strategy employed. Its proof relies on a well-founded relation $\prec_{bu}$ such that every rule R of $\mathscr{E}$-KM is *decreasing w.r.t.* $\prec_{bu}$: for every application ρ of R with conclusion σ and premise σ', we have $\sigma'\prec_{bu}\sigma$. The main difficulty in defining $\prec_{bu}$ arises from $L\to$, which admits applications with conclusion $\alpha\to\beta,\Gamma\overset{u}{\Rightarrow}\alpha$ and left premise $\alpha\to\beta,\Gamma\overset{b}{\Rightarrow}\alpha\,;\alpha$. The labeling mechanism prevents such configurations from causing loops in backward proof search. So, let σ and σ' be the conclusion and the left premise of an application of rule $L\to$; we stipulate that $\sigma'\prec_{bu}\sigma$, since σ' has label b while σ has label u (thus, b is assigned a lower weight than u). Now, we need to accommodate rules $R\overset{b}{\to}$ and $R^{\Box}_{b}$, which, read bottom-up, switch from b to u. In both cases, the left-hand side of the premises evaluates a new formula; e.g., in the application of rule $R\overset{b}{\to}$ having premises $\alpha,\Gamma,\Box\Theta\overset{u}{\Rightarrow}\psi$ and $\alpha,\Gamma,\Theta\overset{u}{\Rightarrow}\beta$ and conclusion $\Gamma,\Box\Theta\overset{b}{\Rightarrow}\alpha\to\beta\,;\psi$, it holds that $\Gamma\cup\Box\Theta\not\rhd\alpha$ (side condition) and $\Gamma\cup\Box\Theta\cup\{\alpha\}\rhd\alpha$ and $\Gamma\cup\Theta\cup\{\alpha\}\rhd\alpha$ (definition of $\rhd$); so we can exploit the evaluation relation to define $\prec_{bu}$ in these cases. Let $\mathrm{Ev}(\Gamma\overset{l}{\Rightarrow}\delta\,;\psi) = \{\varphi\mid\varphi\in\mathrm{Sf}(\Gamma\overset{l}{\Rightarrow}\delta\,;\psi)$ and $\Gamma\rhd\varphi\}$ where $\mathrm{Sf}(\Gamma\overset{l}{\Rightarrow}\delta\,;\psi)$ is the set of the subformulas of the formulas in $\Gamma\cup\{\delta,\psi\}$. Note that $\mathrm{Ev}(\sigma)\subseteq\mathrm{Sf}(\sigma)$. Finally, to treat the other rules we have to take into account the size of a sequents $|\Gamma\overset{l}{\Rightarrow}\delta\,;\psi|$, defined as $|\Gamma|+|\delta|+|\psi|$, where formula size is symbol count, and set size is the sum of its formula sizes. Summarizing: $\sigma'\prec_{bu}\sigma$ iff one of the following conditions holds:

(a) $\mathrm{Sf}(\sigma')\subset\mathrm{Sf}(\sigma)$;

(b) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma')\supset\mathrm{Ev}(\sigma)$;

(c) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ and $\mathrm{label}(\sigma') = \mathrm{b}$ and $\mathrm{label}(\sigma) = \mathrm{u}$;

(d) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ and $\mathrm{label}(\sigma') = \mathrm{label}(\sigma)$ and $|\sigma'|<|\sigma|$.

It is easy to check that $\prec_{\mathrm{bu}}$ is a well-founded relation; verifying that every rule of the calculus $\mathscr{E}$-KM is decreasing w.r.t. $\prec_{\mathrm{bu}}$ is a technical matter.

As for Completeness (point (iv) of the theorem), we prove it by showing that, whenever a sequent σ is not provable in $\mathscr{E}$-KM, a Kripke model witnessing that α does not belong to the logic can be extracted from a failed proof search. To complement our theoretical results with an applied contribution, we have implemented the proof search procedure in the Java framework JTabWB [7].

In the literature, some calculi for KM have been proposed. The sequent calculus introduced in [3] is not terminating and is therefore unsuitable for proof search. The calculus presented in [2] is multi-succedent and introduces two types of implication. Note that the presence of multiple formulas on the right-hand side increases the nondeterminism, and hence the complexity, of proof search. The system proposed in [10] is based on the terminating multi-succedent sequent calculus G4i for intuitionistic propositional logic [4] and lacks the subformula property.

We leave as future work the investigation of cut-admissibility for $\mathscr{E}$-KM; this is a rather tricky task since labels impose strict constraints on the shape of derivations. We also plan to investigate whether our calculus can be used to compute uniform interpolants, as done in [10]. The implementation of the calculus is available at https://github.com/ferram/jtabwb_provers/tree/master/km_ekm, together with a detailed document containing the full technical details and proofs.

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The Good News Translation

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Bezhanishvili et al. [2] suggest a translation from *weak positive logic* (WPL) into *tense information logic* (TIL) [3, 4] on lattices. This suggestion is motivated by the observation that disjunction in WPL behaves like a modal operator in a way that resembles the interpretation of the $\langle\inf\rangle$-operator in TIL. In this abstract, we work out this suggestion.

We introduce an extension of TIL and give a translation of the $\bot$-free fragment of WPL into this extension that preserves and reflects consequence. We then extend this result to the full language of WPL by defining a translation relative to a consequence pair.

1 Preliminaries

Weak positive logic (WPL), studied algebraically as the equational theory of bounded lattices, is the $\{\wedge,\vee,\top,\bot\}$-fragment of classical logic without distributivity. In the frame semantics for WPL developed by Bezhanishvili et al. [2], an *L-model* is a bounded lattice $(X,0,1,\curlywedge,\curlyvee)$ equipped with a valuation assigning to each propositional letter a *filter* of $(X,0,1,\curlywedge,\curlyvee)$. The satisfaction clauses for $\wedge$ and $\top$ are the usual ones, while $\mathfrak{M},x\Vdash_W\bot \iff x=1$, i.e. $\bot$ holds precisely at the top element, and

$$\mathfrak{M},x\Vdash_W \varphi\vee\psi \iff \exists y,z\in X:\ \mathfrak{M},y\Vdash_W\varphi,\ \mathfrak{M},z\Vdash_W\psi,\ y\curlywedge z\preceq x.$$

The non-classical clause for disjunction owes to non-distributivity, and suggests a modal interpretation of non-distributive disjunction.

Tense information logic (TIL), introduced by van Benthem [1] and recently axiomatized and shown to be decidable [3, 4], extends modal information logic (with its supremum modality $\langle\sup\rangle$) by an additional infimum modality $\langle\inf\rangle$. A *lattice model* for TIL is defined as an L-model, but with an arbitrary valuation $V:\mathbf{P}\to\mathscr{P}(X)$ instead of a filter valuation. The semantics of the two modalities are:

$$\begin{aligned}\mathfrak{M},x\Vdash_T\langle\sup\rangle\varphi\psi &\iff \exists y,z:\ \mathfrak{M},y\Vdash_T\varphi,\ \mathfrak{M},z\Vdash_T\psi,\ x=\sup\{y,z\},\\ \mathfrak{M},x\Vdash_T\langle\inf\rangle\varphi\psi &\iff \exists y,z:\ \mathfrak{M},y\Vdash_T\varphi,\ \mathfrak{M},z\Vdash_T\psi,\ x=\inf\{y,z\}.\end{aligned}$$

Bezhanishvili et al. [2] observe a close match between the clause for $\langle\inf\rangle$ and that for WPL-disjunction. The only difference is that for $\varphi\vee\psi$ to hold at a state x in an L-model, it suffices that the meet of states witnessing φ and ψ lies *below* x. Based on this observation, Bezhanishvili et al. propose translating disjunction as follows:

$$T(\varphi\vee\psi) := \mathrm{P}(\langle\inf\rangle T(\varphi)T(\psi)),$$

where P refers to the past-looking diamond, definable as $\mathrm{P}\varphi:=\langle\sup\rangle\varphi\top$. We clarify this modally flavoured aspect of WPL by working out a translation in full detail. We call it *the good news translation*.

2 The $\bot$-Free Translation

Despite initial appearances, making the translation idea from [2] precise turns out to be intricate and, we think, illuminating. In particular, we must address two limitations of the expressive power of TIL. First, in the frame semantics of WPL, propositional letters are interpreted as filters. Accordingly, we aim to define a translation that sends each propositional letter to a formula whose satisfaction set is closed under arbitrary finite meet.

The basic language of TIL can only express *binary* meet by means of the operator $\langle\inf\rangle$. We therefore enrich the language with a *finite-meet* modality $\langle\inf^*\rangle$:

$$\mathfrak{M}, x \Vdash_T \langle\inf^*\rangle\varphi \iff \exists y_1, \ldots, y_n\ (n \geq 1) \text{ with } y_i \Vdash \varphi,\ x = \inf\{y_i\}.$$

We call the resulting language $\mathscr{L}_{T*}$ and the corresponding consequence relation *TIL**. This enriched language is strictly more expressive than the original language of TIL, but it still cannot define the top element of a lattice. Because $\bot$ holds only at the top element in the semantics of WPL, preserving and reflecting consequence would require its translation to characterize that element. As a consequence, we initially only obtain a translation of the $\bot$-free fragment of WPL, which we denote by $\mathscr{L}_W^-$.

Using the enriched language $\mathscr{L}_{T*}$, we can now give a complete definition of the translation of the $\bot$-free fragment of WPL:

$$\begin{aligned} T(p) &= \mathrm{P}(\langle\inf^*\rangle p) \vee \neg\mathrm{FP}\,p, & T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi), \\ T(\top) &= \neg\bot, & T(\varphi \vee \psi) &= \mathrm{P}(\langle\inf\rangle T(\varphi) T(\psi)). \end{aligned}$$

Each TIL valuation V then induces a WPL filter-valuation $\rho V(p) := [\![T(p)]\!]$: the first disjunct of $T(p)$ generates the smallest filter containing $V(p)$ when $V(p) \neq \varnothing$, while the second disjunct ensures non-emptiness when $V(p) = \varnothing$, in which case $\rho V(p) = X$.

Using induction, we can then prove that this is a full and faithful translation:

Theorem 1. *For all* $\varphi, \psi \in \mathscr{L}_W^-$: $\quad \varphi \Vdash_W \psi \iff T(\varphi) \Vdash_T T(\psi)$.

3 Translating $\bot$ Relative to a Consequence Pair

To extend the translation to the full language of WPL, we relativize it to a consequence pair (φ, ψ), obtaining a translation $T_{\varphi,\psi}$. This allows the translation of $\bot$ to depend on the finitely many propositional letters occurring in φ and ψ:

$$T_{\varphi,\psi}(\bot) \;=\; \bigwedge_{p \in \mathrm{Prop}(\varphi,\psi) \cup \{q\}} \mathrm{P}\langle\inf^*\rangle\big(\neg\mathrm{P}\,p \to \neg\mathrm{FP}\,p\big),$$

where q is a fresh propositional letter not occurring in φ or ψ. The basic building block of this translation is the scheme $\neg\mathrm{P}\,p \to \neg\mathrm{FP}\,p$. For every lattice L, every valuation V, and every propositional letter p, the scheme is satisfied at the top element 1 of L. By contrast, at non-top points it need not hold (i.e. it can fail for suitable valuations). Thus, conjoining instances of this scheme over the propositional letters occurring in the consequence pair φ, ψ provides a first approximation of "top-like" behaviour. Indeed, we can prove that for all subformulas χ of φ or ψ: if a point satisfies $T_{\varphi,\psi}(\bot)$, then it satisfies $T_{\varphi,\psi}(\chi)$.

However, $[\![\neg\mathrm{P}\,p \to \neg\mathrm{FP}\,p]\!]$ need not be closed under meet, so it does not in general form a filter — an issue for transforming lattice models into L-models, where truth sets of formulas always are filters. Therefore we apply $\mathrm{P}\langle\inf^*\rangle$ to each instance of the scheme. Finally, the somewhat peculiar fresh letter q

is used in the converse transformation of L-models into lattice models to ensure that $T_{\varphi,\psi}(\bot)$ holds only at the top element. There, we use that q does not occur in φ or ψ, so we may without loss of generality assume that $V(q) = \{1\}$.

Theorem 2. *For every pair of formulas* $\varphi, \psi \in \mathscr{L}_W$: $\quad \varphi \Vdash_W \psi \iff T_{\varphi,\psi}(\varphi) \Vdash_T T_{\varphi,\psi}(\psi)$.

The right-to-left direction is the more involved of the two. For it, we employ a construction that turns a TIL* lattice countermodel L into a WPL countermodel by defining an equivalence relation on L that identifies all worlds satisfying the translation of $\bot$:

$$x \sim y \iff \big(x \Vdash_T T_{\varphi,\psi}(\bot) \text{ and } y \Vdash_T T_{\varphi,\psi}(\bot)\big) \text{ or } x = y,$$

collapsing all worlds where the translation of $\bot$ holds into a single class. In general, $\sim$ is *not* a congruence relation. Instead we define meet and join on $L/\!\sim$ explicitly:

$$|x| \curlyvee_\sim |y| := |x \curlyvee y|, \qquad |x| \curlywedge_\sim |y| := \begin{cases} |x \curlywedge y|, & |x| \neq |1| \neq |y|, \\ |x|, & |y| = |1|, \\ |y|, & |x| = |1|, \end{cases}$$

and verify directly that these operations make $L/\!\sim$ into a bounded lattice with top $|1|$ and bottom $|0|$. With the filter valuation $\rho' V(p) := \{|x| \mid (L,V), x \Vdash_T T_{\varphi,\psi}(p)\}$, an induction (using an auxiliary lemma showing that any point satisfying $T_{\varphi,\psi}(\bot)$ satisfies $T_{\varphi,\psi}(\chi)$ for every subformula χ of the consequence pair) yields:

$$(L,V), y \Vdash_T T_{\varphi,\psi}(\chi) \iff (L/\!\sim, \rho' V), |y| \Vdash_W \chi.$$

This delivers the desired countermodel for the right-to-left direction of Theorem 2.

The translation above introduce two ideas that may be of independent interest: translating relative to a consequence pair, as, in our case, to handle $\bot$, and constructing a countermodel that uses an equivalence relation that need not be a congruence, but whereupon a suitable lattice structure still can be defined.

Together, the two translations make the suggestion of [2] precise; however, importantly, not by translating directly into TIL, as originally proposed, but by passing through a PDL-like extension of TIL, as well as making the translation relative to consequence pairs.

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Local Semantics for Belief

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We present a doxastic model along with a local semantics. On these models, the accessibility relation is defined directly between local states. The global states considered by an agent are now merely collections of local states that the agent considers possible. This results in an interesting three-valued semantics that superficially resembles the three-valued semantics on impure simplicial complexes, but comes with radically different validities.

A new branch in distributed computing concerned with simplicial semantics for epistemic logics has been widely explored as an alternative to relational semantics (see e.g. [10, 8, 5, 2]). Simplicial models shift the focus from worlds (*global states*) towards the local perspectives (*local states*) of the individual agents. Global states are not primitive entities, but combinations of compatible local states. Nevertheless, local states remain subordinate: accessibility relations are defined on the global level. Considering a setting where agents reason less about the environment and more about each other's local knowledge and perspectives, we desire a model in which local states are the central objects. To this end, we propose a relational model with an accessibility relation defined between local states.

We expect this shift in focus to be both helpful and intuitive in the modeling of information dynamics, for instance, when characterizing *faulty* agents. Arbitrary byzantine behaviour has been formalized in the runs-and-systems framework [6]; in simplicial models, existing work mainly focuses on agents that can crash (see e.g. [10, 7]). On simplicial models, a realistic formalization of dynamics involving arbitrary faults requires a notion of belief [3]. Previous proposals interpret belief on the global level [5, 1, 4].

By employing local semantics (a.k.a. *face-semantics*) on simplicial models [10, 5], formulas can be evaluated on vertices, as well as any properly colored set of vertices. However, the standard representation of a distributed system (cf. [9]) assumes that only sets of mutually compatible local states can constitute a global state. Consider, for example, causal relations between local states due to previous events: if Alice's local state a_1 receives a message from Bob's local state b_1, then a_1 is theoretically incompatible with any alternative local state b_2 of Bob, that did not send any message to Alice. Thus, one cannot simply take any collection of local states to be a global state. In simplicial models, facets are indeed maximal sets of compatible local states.

In our *local belief models*, which take inspiration from simplicial models, this compatibility between local states is instead taken care of by a binary relation R, which is assumed to be reflexive: every local state is compatible with itself. A relation $(w_1, w_2) \in R$ indicates the compatibility of w_1 with w_2, from the perspective of w_1. We define the languages and the models as follows.

We fix a finite set of agents Ag and for each agent $a \in \mathsf{Ag}$, a countable set of atomic propositions Prop_a. Assume that $\mathsf{Prop}_a \neq \varnothing$ for all $a \in \mathsf{Ag}$ and that $\mathsf{Prop}_a \cap \mathsf{Prop}_b = \varnothing$ if $a \neq b$. We define a language Fml, as well as a family of languages $\{\mathsf{Fml}_a \mid a \in \mathsf{Ag}\}$, by the following grammars:

$$\begin{array}{ll} \mathsf{Fml} & \psi ::= p_a \mid \neg\psi \mid \psi \wedge \psi \mid \Box_a \psi \qquad (a \in \mathsf{Ag}, p_a \in \mathsf{Prop}_a) \\ \mathsf{Fml}_a & \varphi ::= p_a \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \psi \qquad (p_a \in \mathsf{Prop}_a \text{ and } \psi \in \mathsf{Fml}) \end{array}$$

The formulas in Fml_a (also referred to as *a-formulas*) describe the "perspective" of agent a. We will see in our models that the formulas defined at a-vertices will always be in Fml_a.

Definition (Local Belief Model)**.** A *local belief model* (or LBM) is a structure $\mathscr{M} = (W, R, \chi, v)$ where

1. W is a set of local states;
2. $R \subseteq W \times W$ is a reflexive binary relation on W, i.e. $(s,s) \in R$ for all $s \in W$;
3. χ is a *proper coloring*, i.e. $\chi : W \to \mathsf{Ag}$ such that for all $s \neq t$: $(s,t) \in R$ implies $\chi(s) \neq \chi(t)$;
4. v is a *local valuation*, i.e. $v : W \to \bigcup_{a \in \mathsf{Ag}} \mathscr{P}(()\mathsf{Prop}_a)$ such that $\chi(w) = a$ implies $v(w) \subseteq \mathsf{Prop}_a$.

We call a state s with $\chi(s) = a$ an a-state. Given a set X of states, we define $\chi(X) = \{\chi(s) \mid s \in X\}$. If $|\chi(X)| = |X|$ and $a \in \chi(X)$ for some $a \in \mathsf{Ag}$, then X_a denotes the unique element $s \in X$ with $\chi(s) = a$.

An additional feature of our semantics is the evaluation of formulas on sets of "incompatible" local states: as long as agent Alice considers her state compatible with both Bob's and Charles' local states, it is reasonable to assume that Alice does not consider the mutual incompatibility of Bob's and Charles' states. This is reflected by the extension of R to arbitrarily properly colored sets. Given the set

$$G(\mathscr{M}) = \{X \in \mathscr{P}(()W) \mid X \neq \varnothing \text{ and for all } s,t \in X : s \neq t \text{ implies } \chi(s) \neq \chi(t)\}$$

of all *properly colored* sets (*worlds*) of vertices in $\mathscr{M}$, we define the extension $\tilde{R} \subseteq W \times G(\mathscr{M})$ of R by

$$\tilde{R}(s,X) \quad \Leftrightarrow \quad R(s,t) \text{ for all } t \in X.$$

In local state a_1, Alice now considers possible any combination X of local states for which $a_1 \tilde{R} X$ holds. A notion of belief based on local accessibility relations is arguably more intuitive: Alice's beliefs with respect to Bob's local state can be formed without taking into account Charles' local state. Hence, incorrect beliefs concerning one agent do not influence Alice's beliefs concerning others.

An agent might only consider incomplete worlds: suppose that Alice received inconsistent information from Bob, after which she does not consider any of his local states. In that case, Bob is undefined in Alice's global states. Indeed, our local semantics is three-valued. In fact, it resembles the three-valued semantics on impure simplicial complexes in [10].

As in [10], we only evaluate a formula φ in a world X if φ is *defined* at X.

Definition. Let $\mathscr{M} = (W, R, \chi, v)$ be a local belief model and let $X \in G(\mathscr{M})$. Define the definability relation $\bowtie$ and truth relation $\models$ inductively:

$$\begin{array}{lll} \mathscr{M}, X \bowtie p_a & \text{iff} & a \in \chi(X) \\ \mathscr{M}, X \bowtie \neg\varphi & \text{iff} & \mathscr{M}, X \bowtie \varphi \\ \mathscr{M}, X \bowtie \varphi \wedge \psi & \text{iff} & \mathscr{M}, X \bowtie \varphi \text{ and } \mathscr{M}, X \bowtie \psi \\ \mathscr{M}, X \bowtie \Box_a \varphi & \text{iff} & a \in \chi(X) \text{ and } \mathscr{M}, Y \bowtie \varphi \text{ for some } Y \in G(\mathscr{M}) \text{ with } X_a \tilde{R} Y. \end{array}$$

$$\begin{array}{lll} \mathscr{M}, X \models p_a & \text{iff} & a \in \chi(X) \text{ and } p_a \in v(X_a) \\ \mathscr{M}, X \models \neg\varphi & \text{iff} & \mathscr{M}, X \bowtie \neg\varphi \text{ and } \mathscr{M}, X \not\models \varphi \\ \mathscr{M}, X \models \varphi \wedge \psi & \text{iff} & \mathscr{M}, X \models \varphi \text{ and } \mathscr{M}, X \models \psi \\ \mathscr{M}, X \models \Box_a \varphi & \text{iff} & \mathscr{M}, X \bowtie \Box_a \varphi \text{ and } \mathscr{M}, Y \models \varphi \text{ for all } Y \in G(\mathscr{M}) \text{ with } X_a \tilde{R} Y \text{ and } \mathscr{M}, Y \bowtie \varphi. \end{array}$$

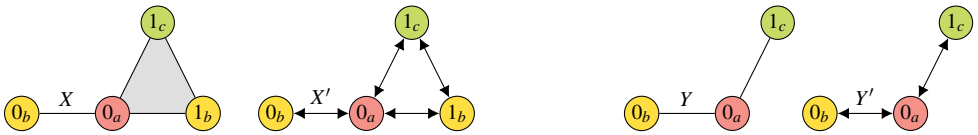
A formula φ is valid, written $\models \varphi$, iff for any $\mathscr{M}$ and any $X \in G(\mathscr{M})$: $\mathscr{M}, X \bowtie \varphi \Rightarrow \mathscr{M}, X \models \varphi$.

Our different treatment of global states and the different interpretation of the absence of agents results in validities different from those in [10], despite similar semantics and a similar notion of validity. Note, in particular, that the axiom $\Box_a\varphi \to \varphi$ is not a validity in our logic (Table 1). Furthermore, not every impure simplicial model (ISM) can be represented as a local belief model, and vice versa. One way to compare the two, however, is by representing the edges in a simplicial model as bidirectional arrows in a LBM structure. In Figure 6, we compare the simplicial semantics on ISMs with our semantics on their natural LBM analogs by analyzing instances of axiom A2.

Conjecture. *The axiom system in Table 1 is sound and complete w.r.t. local belief models.*

Standard:	**Non-standard:**	
(Taut) Prop. tautologies	(ExtLoc)* $\varphi_a \leftrightarrow \Box_a\varphi_a$	for $\varphi_a \in \mathsf{Fml}_a$.
(D)* $\Box_a\varphi \to \neg\Box_a\neg\varphi$	(K$^{\bowtie}$) $\Box_a(\varphi\to\psi)\to(\Box_a\varphi\to\Box_a\psi)$	where $\psi, \Box_a\varphi \ltimes \varphi$.
(4) $\Box_a\varphi \to \Box_a\Box_a\varphi$	(K$\hat{\text{K}}$) $\Box_a(\varphi\to\psi)\to(\Box_a\varphi\to\Diamond_a\psi)$	
(5) $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$	(A1)* $\Box_a(\varphi\to\psi)\to\Box_a\neg\varphi\vee\Box_a\psi$	where $\mathsf{AG}(\varphi)\cap\mathsf{AG}(\psi)=\varnothing$.
(N) From φ, infer $\Box_a\varphi$	(A2)* $\Box_a(\varphi_1\wedge\ldots\wedge\varphi_n)\leftrightarrow\Box_a\varphi_1\wedge\ldots\wedge\Box_a\varphi_n$	where for all $i\neq j$: $\mathsf{AG}(\varphi_i)\cap\mathsf{AG}(\varphi_j)=\varnothing$.
	(Refl)* $\Box_a\psi\wedge\varphi_a\leftrightarrow\Box_a(\psi\wedge\varphi_a)$	for $\varphi_a\in\mathsf{Fml}_a$.
	(MP$^{\bowtie}$) From $\varphi\to\psi$ and φ, infer ψ	where $\psi\ltimes\varphi$.

Table 1: Candidate axiomatization. The symbol $\ltimes$ denotes the *definability consequence* relation as defined in [10, Def. 2.10]. For any formula φ, we define the *agents of* φ as $\mathsf{Ag}(\varphi) := \{a \in \mathsf{Ag} \mid \varphi \ltimes p_a\}$. Axioms annotated with an asterisk do not have obvious analogs in the axiom system in [10, Def. 3.3].



(a) $\mathbb{M}_1$ (ISM, left) and $\mathscr{M}_1$ (LBM) (b) $\mathbb{M}_2$ (ISM, left) and $\mathscr{M}_2$ (LBM)

Figure 6: Two ISMs and corresponding LBM analogs (reflexive arrows omitted). For each agent i, labeling 1_i (resp. 0_i) indicates p_i (resp. $\neg p_i$). In (a), the formula $\Box_a(p_b\wedge p_c)\to\Box_a p_b\wedge\Box_a p_c$ is defined at the *ab*-edges X of $\mathbb{M}_1$ and X' of $\mathscr{M}_1$. It is false at $(\mathbb{M}_1, X)$, because $\mathbb{M}_1, X \models \Box_a(p_b\wedge p_c)$ but $\mathbb{M}_1, X \models \neg\Box_a p_b$, whereas it is true at $(\mathscr{M}_1, X')$, because $\mathscr{M}_1, X' \models \neg\Box_a(p_b\wedge p_c)$ as $p_b\wedge p_c$ is false at the world $\{0_a, 0_b, 1_c\} \in G(\mathscr{M}_1)$. In (b), the formula $\Box_a\neg p_b\wedge\Box_a p_c\to\Box_a(\neg p_b\wedge p_c)$ is undefined at $(\mathbb{M}_2, Y)$, because $\mathbb{M}_2, Y \not\models \Box_a(\neg p_b\wedge p_c)$, whereas it holds at $(\mathscr{M}_2, Y')$. To see that $\mathscr{M}_2, Y' \models \Box_a(\neg p_b\wedge p_c)$, we consider the only world $\{0_a, 0_b, 1_c\} \in G(\mathscr{M}_2)$ $\tilde{R}$-accessible from Y'_a where $\neg p_b\wedge p_c$ is defined.

The local belief model is best suited for scenarios that do not involve restrictions (that the agents are aware of) on global states, such that the agents consider any combination of local states compatible. In distributed computing, one setting without such restrictions is *binary consensus* (see e.g. [9]). We find a similar setting in Social Network Logic (see e.g. [11, 12]), when we represent individual opinions using local states. Consider the following scenario, formalized in Figure 7:

In the upcoming elections, Alice, Bob, and Charles will each be voting either progressively or conservatively. Alice is progressive, but keeps her preference private. Bob is publicly progressive. Charles, secretly conservative, convinced Bob that he is also progressive. Alice has received inconsistent information with respect to Charles and does not know what to believe anymore.

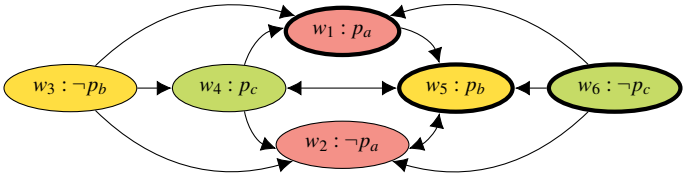


Figure 7: The model $\mathscr{M}$. A progressive (resp. conservative) local state of agent i has valuation p_i (resp. $\neg p_i$). Actual local states are highlighted by a thicker border. Reflexive arrows are omitted.

Note that w_3 has no incoming arrows: none of the a- and c-states believe that Bob is conservative. Bob's false belief concerning Charles is valid on the model: $\mathscr{M} \models \mathscr{B}_b p_c$. In contrast, Alice doesn't consider any c-states at all (in either a-state). For example, the only maximal world considered by a-state w_1 is $\{w_1, w_5\}$. As $\mathscr{M}, \{w_1, w_5\} \models \mathscr{B}_b \Diamond_c \neg p_a$, we get $\mathscr{M}, \{w_1\} \models \Box_a \mathscr{B}_b \Diamond_c \neg p_a$.

We presented a framework that is closely related to simplicial models, but defines a local interpretation of belief accompanied by a fully local semantics. Additionally, we presented a candidate axiomatization. We are currently preparing a paper containing a completeness proof via canonical models for the proposed axiom system. In future work, we plan to extend the logic with dynamics of various types of announcements, in order to model lying and general Byzantine behaviour.

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Logics for Modelling and Reasoning About Concurrent Multiplayer Games with Admissibility of Action Profiles

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1 Introduction

Strategic reasoning about capabilities of agents and coalitions to plan and implement strategies to reach their objectives, or to prevent others from reaching theirs, is an essential aspect of multi-agent systems (MASs). *Concurrent game models* (CGMs, [1, 6]) are commonly used models to formally capture the game-theoretic aspects of strategic interactions in MASs, by endowing each system state with a strategic game form assigning an outcome state to each *action profile* composed of individually available actions of all agents. CGMs are used as models for some of the most prevalent logics for strategic reasoning, including Coalition Logic (CL, [6]) and Alternating-time Temporal Logic (ATL, ATL*, [1]). These two logics feature a *strategic operator* $\langle\langle C\rangle\rangle\varphi$, which intuitively means that 'coalition C has a joint strategy to realize goal φ *regardless of* any other agents' actions', or simply 'C can enforce goal φ'.

A standard assumption in CGMs is *agents' independence*, meaning that any action profile composed of autonomously and independently chosen individual actions currently available to each respective agent is executable. However, this assumption is not so common in reality, where individual actions are often *conflicting* and *cannot co-occur*. One natural source of such restrictions is purely *physical* constraints. For instance, two agents cannot occupy the same space at the same time; two users cannot simultaneously use the same printer, tool, or other physical resources. Another common source of such restrictions is *normative* constraints (cf. [7, 2]) that impose natural restrictions on the collective behaviour of all agents, typically by specifying obligations or prohibitions, like "one should listen when someone else is talking".

In order to develop variants of CGMs where not all action profiles are admissible, one has to design mechanisms handling the *non-admissible action profiles* that may arise when agents choose actions individually. For this we propose three natural approaches. The first two allow agents to act independently and choose non-admissible action profiles, whose outcomes are respectively the *current state* and a *deadlock state*. The third approach requires that at each round, agents choose actions *sequentially* in some priority order, which are then concurrently executed after everyone has made their choices. Importantly, agents cannot choose actions that conflict with those already chosen by agents higher in the order. Thus, it is not possible to end up with a non-admissible action profile, but agents no longer act independently.

These approaches are formalized by variants of CGMs, all of which can be transformed into standard CGMs and *vice versa*. We design suitable logics – conservative extensions of ATL* and ATL – to reason about these CGM variants, capturing the strategic abilities to exploit or ward off exploitation of mechanisms for dealing with non-admissible action profiles. We explore the technical properties of these logics, for most of which we establish the complexity of their model checking (MC) and satisfiability (SAT) problems, and propose sound and complete tableaux and axiomatic systems.

2 Preliminaries

We fix a finite set Agt of **agents** and a countable set AP of **atomic propositions**. Subsets of Agt will be called **coalitions**. A **concurrent game model** (CGM) is a tuple $\mathscr{M} = (\mathsf{S}, \mathsf{Act}, \mathfrak{g}, V)$, where

- S is a non-empty set of **states**, Act is a non-empty set of **actions**, $V : \mathsf{AP} \to \mathscr{P}(S)$ is a **valuation**.
- $\mathfrak{g}$ is a **game map** sending every state $w \in S$ to a **strategic game form** $\mathfrak{g}(w) = (\mathsf{act}_w, \mathsf{out}_w)$ over S, where $\mathsf{act}_w : \mathsf{Agt} \to \mathscr{P}^+(\mathsf{Act})$ is a function assigning to each agent $\mathsf{a} \in \mathsf{Agt}$ a non-empty set of **actions available to** her, and $\mathsf{out}_w : \prod_{a \in \mathsf{Agt}} \mathsf{act}_w(\mathsf{a}) \to \mathsf{S}$ is a function assigning an **outcome** in S to each **action profile** $\zeta \in \prod_{\mathsf{a} \in \mathsf{Agt}} \mathsf{act}_w(\mathsf{a})$.

A **joint action for** $\mathrm{C} \subseteq \mathsf{Agt}$ **at** $w \in \mathsf{S}$ is a tuple of individual actions $\zeta_{\mathrm{C}} \in \prod_{\mathsf{a} \in \mathrm{C}} \mathsf{act}_w(\mathsf{a})$.

3 Approach 1: Blocking

This approach models prevalent scenarios where non-admissible action profiles only stop the system from making progress, with the agents forced to revise their choices, until an admissible action profile is collectively selected.

Example 1 (Pavement tango)**.** This is a commonly occurring real-life situation. Two pedestrians are walking towards each other on a narrow pavement, wide just enough so they can pass each other only if moving to its opposite edges. As they approach each other, each pedestrian has three available actions: walk straight, i.e. stay in the middle, or step to their left or right. If they both step to their left or right, then they can walk past each other smoothly; otherwise, they will stop before they collide with each other and keep changing their choices of actions, until they eventually coordinate on both going left or right.

This approach is **formalized** by **CGM with blocking** ($\mathrm{CGM_B}$), a variant of CGM where each state is assigned a set of admissible action profiles and the outcome function assigns the current state to each non-admissible action profile.

The **language** of ATL^* **with Blocking** ($\mathsf{ATL}^*_\mathsf{B}$) extends that of ATL^* by adding to state formulas strategic operators $\langle\langle \mathrm{C} \rangle\rangle \mathscr{B}$ and $\langle\langle \mathrm{C} \rangle\rangle \overline{\mathscr{B}}$ stating abilities to respectively enforce and prevent blocking. The **semantic clause** for $\langle\langle \mathrm{C} \rangle\rangle \mathscr{B}$ is that there is a joint action for C such that every action profile extending it is non-admissible, and similarly for $\langle\langle \mathrm{C} \rangle\rangle \overline{\mathscr{B}}$. ATL_B is the state formula fragment of $\mathsf{ATL}^*_\mathsf{B}$.

Technical results:

1. The complexity of *MC* and *SAT* for ATL_B and $\mathsf{ATL}^*_\mathsf{B}$ is the same as that for ATL and ATL^* respectively. The MC complexity is established by reduction to $\mathsf{ATL}/\mathsf{ATL}^*$ model checking and straightforward algorithms for model checking $\langle\langle \mathrm{C} \rangle\rangle \mathscr{B}$ and $\langle\langle \mathrm{C} \rangle\rangle \overline{\mathscr{B}}$, while the SAT complexity is established by non-trivial adaptations of the tableaux-based methods for deciding ATL ([4]) and ATL^* ([3]).
2. Sound and complete *tableaux* are proposed for ATL_B and $\mathsf{ATL}^*_\mathsf{B}$.
3. *Axiomatization:* The extension of the axiomatic system for ATL in [5] with the following axioms for the *blocking modalities* is sound and complete:
   1. **Locality**: $\varphi \wedge \langle\langle \mathrm{C} \rangle\rangle \mathscr{B} \to \langle\langle \mathrm{C} \rangle\rangle \mathsf{X} \varphi$.
   2. **Duality**: $\langle\langle \mathrm{C} \rangle\rangle \mathscr{B} \to \neg \langle\langle \overline{\mathrm{C}} \rangle\rangle \overline{\mathscr{B}}$.
   3. **Progression**: $\langle\langle \mathrm{C} \rangle\rangle \mathsf{X} \varphi \to \varphi \vee \langle\langle \mathrm{C} \rangle\rangle \overline{\mathscr{B}}$.
   4. **Coalition monotonicity for blocking**: $\langle\langle \mathrm{C}_1 \rangle\rangle \mathscr{B} \to \langle\langle \mathrm{C}_2 \rangle\rangle \mathscr{B}$, where $\mathrm{C}_1 \subseteq \mathrm{C}_2$.

4 Approach 2: Deadlocking

This approach rests on a similar idea, but models scenarios where non-admissible action profiles lead to catastrophic and irreversible consequences, modelled by *deadlock states* from which the MAS can never leave. It *incentivizes* choosing admissible action profiles by *deterrence* of deadlocking, which is the idea behind the *Mutually Assured Destruction (MAD)* doctrine for preventing the use of nuclear weapons.

Example 2 (Chicken game)**.** Two cars are racing against each other at a high speed on a narrow road, wide just enough so they can pass each other only if swerving to its opposite edges. The driver of each car has three available actions: drive straight, i.e. in the middle, or swerve to their left or right. If they both swerve to the left or right, then they can drive past each other smoothly. These are the admissible action profiles. Otherwise, they will crash into each other, which will be the end of the story.

This approach is **formalized** by **CGM with deadlocking** ($\mathrm{CGM_D}$), which is akin to $\mathrm{CGM_B}$ but contains a set of *designated deadlock states*, and the outcome function sends a non-admissible action profile to a *deadlock state*, where the only transition leads to itself.

The **language** of ATL^* **with Deadlocking** ($\mathsf{ATL}^*_\mathsf{D}$) extends ATL^* with a state atom n_δ for the designated deadlock states, with **semantics** $\mathscr{M},w \models n_\delta$ iff w is a deadlock state. ATL_D denotes the state formula fragment of $\mathsf{ATL}^*_\mathsf{D}$.

Technical results:

1. The complexity of *MC* and *SAT* for ATL_D/$\mathsf{ATL}^*_\mathsf{D}$ is the same as that for ATL/ATL^*, shown by similar methods as those for ATL_B/$\mathsf{ATL}^*_\mathsf{B}$ (see above).
2. Sound and complete *tableaux* are proposed for ATL_D and $\mathsf{ATL}^*_\mathsf{D}$.
3. *Axiomatization:* The extension of the axiomatic system for ATL in [5] with the following axiom for n_δ is sound and complete: **Locality**: $(n_\delta \wedge \varphi) \to \langle\langle\varnothing\rangle\rangle \mathsf{X}(n_\delta \wedge \varphi)$

5 Approach 3: Sequential choices

At each state, the agents no longer choose actions concurrently and independently, but do so *sequentially* in some linear priority order. When it is an agent's turn to select her action, she may only choose a *'safe'* action, i.e. one that is compatible with those already chosen, in the sense of enabling at least one *admissible action profile* to be eventually selected. The *crux* is that the MAS can preclude non-admissible action profiles by restricting agents' choices, and thus preventing them from choosing 'unsafe' actions that would necessarily lead to non-admissible action profiles. *Examples* of this approach include crossroads regulated by traffic lights, and the order of speaking at a conference controlled by the chair.

The **formalization** is similar to $\mathrm{CGM_B}$, but the outcome function is *only* defined on the set of admissible action profiles, and the *mechanisms determining the linear priority orders* of choices are included in the model. Depending on the nature of such mechanisms, there are three variations: **CGM with global/local pre-defined hierarchy** ($\mathrm{CGM_{GPH}}$/$\mathrm{CGM_{LPH}}$) and **with scheduler** ($\mathrm{CGM_{SCH}}$). A hierarchy assigns a linear order of agents to each state, whereas a scheduler assigns such an order to each history.

We add to the language of ATL^* state atoms $\mathsf{a} \prec \mathsf{b}$ for agents a and b, meaning that at the current state, a chooses her action before b. The resulting **languages** are $\mathsf{ATL}^*_\mathsf{SCA}$ (ATL^* **with Sequential Choices of Actions**) and its fragment $\mathsf{ATL}_\mathsf{SCA}$. By interpreting $\mathsf{ATL}^*_\mathsf{SCA}$ on the above CGM variants, we obtain ATL^* **with Global Pre-defined Hierarchy** ($\mathsf{ATL}^*_\mathsf{GPH}$), **with Local Pre-defined Hierarchy** ($\mathsf{ATL}^*_\mathsf{LPH}$), **with Scheduler** ($\mathsf{ATL}^*_\mathsf{SCH}$), and similarly for their $\mathsf{ATL}_\mathsf{SCA}$-fragments.

We note that in these logics, the ATL^* strategic operator $\langle\langle \mathrm{C}\rangle\rangle\varphi$ has *non-standard* **semantics**, where strategies are generalized to *choice strategies*, which instruct an agent what to do, given the current

history and the current set of 'safe' actions. Choice strategy profiles are used to induce admissible action profiles and thus paths, by soliciting agents' choices according to the current priority order of agents.

We have obtained the following **technical results**:

1. The *MC complexity* of these logics is the same as their ATL/ATL* bases, proven by a non-trivial reduction to two-player turn-based games with imperfect information.
2. For *validities*,
   (a) Formulas expressing standard properties of linear orders are valid in all these logics: irreflexivity ($\neg a \prec a$), transitivity ($a \prec b \wedge b \prec c \to a \prec c$), and trichotomy ($a \prec b \vee b \prec a$, where $a \neq b$).
   (b) **GlobalHier**: $a \prec b \to \langle\langle \varnothing \rangle\rangle \mathsf{X}\, a \prec b$ is valid in $\mathsf{ATL}_{\mathsf{GPH}}$ but not in $\mathsf{ATL}_{\mathsf{LPH}}$.
   (c) $\mathsf{ATL}_{\mathsf{LPH}}$ ($\mathsf{ATL}^*_{\mathsf{LPH}}$) and $\mathsf{ATL}_{\mathsf{SCH}}$ ($\mathsf{ATL}^*_{\mathsf{SCH}}$) share the same validities.

Conceptually, it is interesting to see how the **priority orders impact strategic abilities**:

1. Choosing *earlier* often gives agents stronger strategic abilities, especially when competing for *exclusive* control over some resources.
2. On the other hand, it is sometimes more advantageous to choose *later*, so that the agent can *react* better to previously chosen actions.

6 Conclusion

In this work we explored three natural approaches for handling non-admissible action profiles representing agents' dependence in CGMs. All these approaches lead to natural variants of CGMs. We proposed extensions of ATL* and ATL to reason about these CGM variants, and investigated computational properties – complexity of the MC and SAT problems – and axiomatization of these logics. For most of these problems we have obtained answers, and we plan to settle the rest in a follow-up journal paper.

Our work has potential for applications in formal verification and synthesis of MASs with natural mechanisms for handling conflicting actions and norm violations, as well as in logical modelling of norms and study of normative MASs. It would also be interesting to extend our logics with epistemic, dependence, or deontic operators.

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Modal Logic in the Middle Ages

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Like logic in general, also *modal logic* was founded by *Aristotle*. The Stagirite developed, besides his theory of *categorical* syllogisms, a theory of *modal* syllogisms which deals with the question whether every (or some, or no) object which falls under a subject term *S necessarily*, or *possibly*, also falls under a predicate term *P*. Otherwise ancient modal logicians stated some elementary laws for the logical relations between the modalities 'necessary', 'possible' and 'impossible' and the "improper" modalities 'true' and 'false', to wit:

MOD 1 $\quad \Box p \Rightarrow p \Rightarrow \Diamond p$

MOD 2 $\quad \neg\Diamond p \Rightarrow \neg p \Rightarrow \neg\Box p.$

In the wake of Aristotle, two notions of *contingency*, have been considered, viz., "one-sided" possibility vs. "two-sided possibility":

DEF 1 $\quad C_1(p) \Leftrightarrow_{\text{df}} \Diamond p$

DEF 2 $\quad C_2(p) \Leftrightarrow_{\text{df}} (\Diamond p \wedge \Diamond\neg p).$

Medieval modal logic differs from its ancient predecessor mainly in two areas. On the one hand, it is always based on some binary modal operators, namely the relations of *strict implication*, of *repugnancy*, and of *compatibility*, where implication is defined, as usual, by

DEF 3 $\quad (p \rightarrow q) \Leftrightarrow_{\text{df}} \neg\Diamond(p \wedge \neg q).$

A *repugnancy* between p and q obtains iff the truth of one proposition implies the falsity of the other:

DEF 4 $\quad R(p,q) \Leftrightarrow_{\text{df}} (p \rightarrow \neg q).$

Propositions p and q are compatible, or *compossible*, iff they are not repugnant. Medieval logicians usually express this by saying that p "stands together" ("stat cum") with q:

DEF 5 $\quad S(p,q) \Leftrightarrow_{\text{df}} \Diamond(p \wedge q).$

On the other hand, and more importantly, medieval logicians systematically investigated the necessary and sufficient conditions for the modal properties of "hypothetical" propositions, i.e., of conjunctions, disjunctions, and conditionals. For this task they presupposed the standard laws which establish the logical interrelations between the modalities 'necessary', 'possible', and 'impossible':

MOD 3 $\quad \Box p \Leftrightarrow \neg\Diamond\neg p$

MOD 3′ $\quad \Diamond p \Leftrightarrow \neg\Box\neg p$

MOD 4 $\quad \neg\Diamond p \Leftrightarrow \Box\neg p.$

Furthermore, medieval logicians made use of the rule that logical entailments *preserve* the modal status so that

PRES 1 If $(p \Rightarrow q)$, then $(\Box p \Rightarrow \Box q)$

PRES 2 If $(p \Rightarrow q)$, then $(\Diamond p \Rightarrow \Diamond q)$.

John Buridan (ca. 1300–1358) expressed these rules succinctly by saying that it is "impossible for what is false to follow from truths or what is impossible from the possible or what is not necessary from what is necessary" [1, p. 77]. Now, since a conjunction trivially entails each of its conjuncts, it follows that if $(p \wedge q)$ is *necessary*, so must be both p and q:

CONJ 1 $\Box(p \wedge q) \Rightarrow \Box p$

CONJ 2 $\Box(p \wedge q) \Rightarrow \Box q$.

Similarly, the *possibility* of the entire conjunction entails the possibility of each conjunct:

CONJ 3 $\Diamond(p \wedge q) \Rightarrow \Diamond p$

CONJ 4 $\Diamond(p \wedge q) \Rightarrow \Diamond q$.

As a matter of fact, CONJ 1, 2 can be summarized and generalized into the equivalence

CONJ 5 $\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q$.

Thus, Paul of Venice (ca. 1370–1429) stated: "For the necessity of an affirmative conjunctive proposition the necessity of each part is necessary and sufficient" [6, p. 132]. On the other hand, as Walter Burley (ca. 1275–1345) recognized, "the *possibility* of each part is *not* enough for the possibility of a conjunction. Rather together with this it is required that all the parts be compossible among themselves" [2, p. 198]. Expressing the compossibility of p and q by means of the relation $R(p, q)$, one obtains:

CONJ 6 $\Diamond(p \wedge q) \Leftrightarrow \Diamond p \wedge \Diamond q \wedge \neg R(p, q)$.

By negating both sides of this equivalence, it follows that a conjunction is *impossible* iff at least one of the disjuncts is impossible or if p and q are repugnant to each other:

CONJ 7 $\neg\Diamond(p \wedge q) \Leftrightarrow \neg\Diamond p \vee \neg\Diamond q \vee R(p, q)$.

As John Dorp (ca. 1370–1418) put it: "For the impossibility of an affirmative conjunction it suffices and is required that one of its main propositional parts be impossible or that one be incompossible with the other" [7, p. 137].

If one adopts the notion of contingency in the sense of C_1, the conditions for the *contingency*$_1$ of a conjunction can be read off from CONJ 6: $C_1(p \wedge q) \leftrightarrow C_1(p) \wedge C_1(q) \wedge \neg R(p, q)$. Since the *compossibility* of p and q presupposes that p and q must be *possible* (simpliciter), this law can be simplified to

CONJ 8 $C_1(p \wedge q) \Leftrightarrow \neg R(p, q)$.

Things become more complicated, however, if one adopts conception C_2. According to Paul of Venice, "For the contingency of an affirmative conjunction, it is required and it suffices that one part be contingent and the other part not be impossible or incompossible with the first" [6, p. 132]:

CONJ 9 $C_2(p \wedge q) \Leftrightarrow (C_2(p) \vee C_2(q)) \wedge \neg R(p, q)$.

Corresponding results for the necessity, possibility, or contingency of a *disjunction* may be obtained by application of the "De Morgan" law $(p \vee q) \Leftrightarrow \neg(\neg p \wedge \neg q)$. In particular, as noted by Burley, "for the possibility of a disjunction it is sufficient and is required that one part of it be possible" [2, p. 205]:

DISJ 1 $\quad \Diamond(p \vee q) \Leftrightarrow (\Diamond p \vee \Diamond q)$.

Accordingly, "for the *impossibility* of a disjunction the impossibility of *both parts* is required" (ibid.). Burley further thought that "for the *necessity* of a disjunction it is sufficient and is required that one part of it be necessary or that its parts contradict each other or are *equipollent* to contradictories" (ibid.):

DISJ 2* $\quad \Box(p \vee q) \Leftrightarrow (\Box p \vee \Box q \vee (p \leftrightarrow \neg q))$.

However, as was discovered by Dorp, the third condition on the right-hand side of this formula has to be weakened to '$(p \leftarrow \neg q)$': "For the necessity of an affirmative disjunction, it suffices and is required that one of its parts is necessary or that the disjunction is composed from parts which participate in the law of contradictory *or subcontrary* propositions" (cf. [7, p. 39]).

The determination of the necessary and sufficient conditions for the contingency$_2$ of a disjunction turned out to be fairly difficult. The law

DISJ 3 $\quad C_2(p \vee q) \Leftrightarrow (C_2 p \vee C_2 q) \wedge \neg R(\neg p, \neg q)$

was only approximately discovered by Paul of Venice and by his pupil Paul of Pergula (ca. 1400–1451). In a first approach, Paul of Venice maintained that for the contingency of disjunction "it is required that both parts be contingent and that no part be repugnant to another or be mutually contradictory" [6, p. 133]. In a later chapter, however, this criterion was improved by adding the condition that *the negations* of the two disjuncts are not repugnant to each other. In a similar way, Paul of Pergula maintained: "A disjunction is contingent if each part is contingent and if neither the parts *nor their contradictories* are repugnant" (cf. [5, p. 22]). However, none of these criteria is fully correct. On the one hand, if only one of the propositions p, q is contingent$_2$ while the other is impossible, say if $C_2 p \wedge \neg\Diamond q$, then the disjunction $(p \vee q)$ remains contingent$_2$. On the other hand, there exist propositions p, q such that p and q are *incompatible* and yet their disjunction is contingent$_2$, e.g., *contrary* propositions such as 'Antichrist is white' and 'Antichrist is black'.

As regards the modal properties of *implications*, many medieval logicians argued that if the implication is *true*, then it is *necessary*, but if it is *false*, then it is *impossible*:

IMPL 1 $\quad (p \to q) \Rightarrow \Box(p \to q)$
IMPL 2 $\quad \neg(p \to q) \Rightarrow \neg\Diamond(p \to q)$.

Interestingly, these hypotheses are provably equivalent to the characteristic axioms of our modern systems **S4** and **S5**. The key for this equivalence consists in a law according to which a proposition p is *necessary* iff p *is* (strictly) *implied* by its own negation:

IMPL 3 $\quad \Box p \Leftrightarrow (\neg p \to p)$.

As a corollary it follows that p is *possible* iff p does *not imply* its own negation:

IMPL 4 $\quad \Diamond p \Leftrightarrow \neg(p \to \neg p)$.

The validity of IMPL 3 and IMPL 4 was recognized, e.g., by Burley and by Buridan. Thus, Burley stated that a presumed law according to which one and the same consequent cannot be implied both by an antecedent p and by p's negation, only holds for *non-necessary* consequents: "I say that the

same consequent does not follow from the same antecedent affirmed and denied, *unless the opposite of that consequent includes contradictories*" [2, p. 160]. Similarly, Buridan remarked, rather incidentally, that "a *possible* proposition never entails its own negation", where the editor of the English translation, G. E. Hughes, hastened to explain: "Note that the principle appealed to is not that no proposition whatsoever can entail its own contradictory, but only that no *possible* proposition can do so" [3, p. 86]. These medieval discoveries are of great importance for modern discussions of so-called *connexive logic* (as defined in [8]), for they show that their characteristic axioms, i.e., "Aristotle's Thesis" and "Boethius's Thesis", only hold in a "humble" form, i.e., when restricted to non-necessary consequents, and to self-consistent antecedents, respectively. A detailed investigation of "Connexive Logic in the Middle Ages" has recently been published in [4].

Now, the characteristic axioms of **S4** and **S5**, $(\Box p \to \Box\Box p)$ and $(\Diamond p \to \Box\Diamond p)$, allow to prove that, as maintained by Paul of Venice, "every conditional which is true is necessary and every conditional which is false is impossible" [6, p. 131], provided that one conceives of a conditional as a strict implication. For if $(p \to q)$ or $(\neg(p \wedge \neg q))$ is *true*, then according to **S4** $\Box(\neg(p \wedge \neg q))$, i.e., $\Box(p \to q)$ is *necessary*; and if $(p \to q)$ is *false*, i.e., if $\Diamond(p \wedge \neg q)$, then according to **S5** $\Box\Diamond(p \wedge \neg q)$, i.e., $(p \to q)$ is *necessarily false*, or impossible. As is easily checked, the modern laws **S4** and **S5** can conversely be derived, by means of NEC and POSS, from the medieval principles $(p \to q) \Rightarrow \Box(p \to q)$ and $\neg(p \to q) \Rightarrow \neg\Diamond(p \to q)$.

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Unification and Admissible Rules for Modal Logics of Irreflexive Trees of Bounded Height

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This paper applies Ghilardi's (2000) method of projective formulas to study the unification problem for the modal logics $\mathrm{K}_n := \mathrm{K} + \Box^n\bot$ (the logics of irreflexive trees of height at most n). Providing a semantic characterization of projective formulas over K_n, we prove that for every n, the logic K_n has finitary unification type, and that both the unification problem and the admissibility problem are decidable for K_n. The proof yields an EXPSPACE upper bound on the computational complexity of the unification problem for K_n. Although the finitary unification type of each K_n was already established by Balbiani et al. (2023), the proof presented here is clearer and more transparent. Finally, building on our characterization of K_n-projective formulas, we adapt Jeřábek's (2005) approach and, for the first time, construct an explicit basis of generalized admissible rules for these logics.

1 Introduction

The *unification problem* for a logic L usually asks whether a given formula φ has an L-unifier; that is, whether there exists a substitution σ such that $\sigma(\varphi)$ is a theorem of L. When φ is unifiable in L, a natural subsequent question is: how large and complex is the set of all L-unifiers for φ? Unification theory formalizes this question by assigning to L a *unification type*; this type classifies the structure of the set of all L-unifiers for any unifiable formula.

The systematic study of unification in propositional logics was pioneered by Silvio Ghilardi in the 1990s (see [5, 6]). By introducing the concepts of projective formulas and projective substitutions, he established that many modal and intermediate logics — including $\mathrm{Int}, \mathrm{K4}, \mathrm{S4}, \mathrm{GL}$, etc. — have finitary unification type, and that a finite complete set of unifiers for any unifiable formula can be effectively constructed. This work yielded simple, uniform proofs of Rybakov's earlier decidability results on the rules admissibility [9, 10]. Moreover, Ghilardi's framework later became essential for obtaining explicit bases of admissible rules, as demonstrated in the work of Iemhoff [7] for intuitionistic logic and Jeřábek [8] for modal logics.

The finitary unification type of the logics $\mathrm{K} + \Box^n\bot$, which we denote by K_n, was established in 2023 by P. Balbiani et al. [3]. Essentially, they extended their combinatorial method from the previously solved case $n = 2$ (see [1]) and, using intricate lemmas, showed that, for all $n \geq 2$, unification in K_n is finitary and not filtering. The present paper provides an alternative and simpler proof of their results, and applies the developed technique to construct, for every natural number n, an explicit basis of K_n-admissible generalized rules.

2 A finitary unification type

The case $n = 1$ is exceptional and straightforward. Indeed, the logic $\mathrm{K} + \Box\bot$ has unitary unification type, because any formula in K_1 is equivalent to some formula without modalities. Hence the unification problem for K_1 can be reduced to the unification problem for classical propositional logic CL, where it is known that all unifiable formulas are projective and have most general unifiers (for the details, see e.g. [2]).

It is easy to see that, for every $n \in \mathbb{N}$, the logic K_n is locally tabular; that is, over any finite set of atomic propositions, there are only finitely many formulas up to provable equivalence. In 2022, W. Dzik et al. [4] studied unification in locally tabular normal modal logics (which need not necessary be transitive) and gave a criterion for such logics to have finitary unification type. Informally, a logic of this kind has finitary unification precisely when, for each substitution σ, there exist maps G_σ and F_σ between Kripke models for L that fulfill five specific conditions. We do not list all the conditions here (see [4] for the full statement), but note that most of them are essentially bisimulation-type requirements. Using this criterion, they proved that some logics defined by simple finite frames have finitary unification type, and they also exhibited an example of a (even tabular) locally tabular logic with nullary unification type.

To establish finitary unification for $\mathrm{K} + \Box^n\bot$ $(n \geq 2)$, we do not follow the complex criterion from [4]; instead, we are going to adapt Ghilardi's framework from [6]. There are two fundamental obstructions to applying Ghilardi's theorem directly to the logics K_n:

(i) K_n are not transitive, for $n \geq 3$;

(ii) Kripke frames for K_n fail to satisfy the extension condition.

We overcome the first issue (i) by modifying Ghilardi's basic substitution θ^A_φ together with the accompanying arguments. As in the transitive case, we prove that a formula φ is K_n-projective iff the class of its K_n-models has the so-called K_n-extension property. The definition of the K_n-extension property is essentially the same as in [6]; although we must now be careful with generated submodels due to the lack of transitivity. The main idea in the proof is as follows: forming the composed substitution $\overline{\theta}$ from all basic substitutions, it suffices to apply $\overline{\theta}$ at most n times to unify any K_n-model. This allows us to bypass the intricate reasoning involving minimal ranks and bisimulation equivalence classes that was required in Ghilardi's transitive setting.

To address the second issue (ii), we verify that the construction of sums (or disjoint unions) of K_n-models of height exactly n is not required, and that K_n-models nevertheless possess the necessary properties. Next, we observe that for the unification problem in K_n, it suffices to consider formulas of modal depth at most n. Combining this observation with the characterization of K_n-projective formulas, we establish that each K_n has finitary unification type. As a corollary, we obtain the decidability of the unification problem for the logics K_n and provide a description of K_n-admissible rules by using a K_n-projective approximation (a slight modification of the projective approximation to our setting).

The complexity analysis shows that each logic K_n has the polysize model property; consequently, it can be shown that its consistency problem is NP-complete. Hence we prove that K_n-projective formulas lie in Π^p_2 (the second level of the polynomial hierarchy) and that the K_n-unification problem belongs to EXPSPACE.

3 Basis of generalized admissible rules

Finally, semantically characterizing K_n-projective formulas, we can adopt Jeřábek's framework [8] and exhibit an explicit basis of generalized K_n-admissible rules. Namely, we prove that the sequences

$$\Box\varphi \to \bigvee_{i<k} \Box\psi_i \ \rhd\ \{\varphi \wedge \Box^{n-1}\bot \to \psi_i \mid i < k\} \tag{$\star_n$}$$

form a basis of K_n-admissible generalized rules. The main difference between ($\star_n$) and the basis for $\mathrm{GL}+\Box^n\bot$ presented in [8] is that we do not place φ under a $\Box$ on the right-hand side, because K_n lacks transitivity.

The proof outline mainly follows the approach of [8]: we introduce an auxiliary semi-normal modal logic $\mathrm{K}_n^{\Box}$ that reflects deducibility in the AR-system axiomatized by ($\star_n$). The next step is to prove soundness and completeness of $\mathrm{K}_n^{\Box}$ w.r.t. the so-called n-extensible K_{n+1}-frames. Ultimately, combining the completeness with our K_n-extension property yields the desired result.

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On the Computability-Theoretic Complexity of Model Checking in S5

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We study the global satisfaction relation of an infinite computable **S5**-model $\mathscr{M}$. Even when $\mathscr{M}$ is computable, this set need not be decidable, and we ask how complicated it can be as a set of natural numbers. We give a complete classification using the Ershov hierarchy, which stratifies sets of natural numbers by the number of mind-changes needed to approximate them computably. We prove that, over computable models, **S5** exhibits a tightly stratified but nontrivial hierarchy of model-checking complexity, invisible in the finite-state setting.

1 Introduction and setup

Over finite Kripke structures, model checking for the basic modal language is decidable in linear time [2, 1]. By contrast, for infinite computable models even the global satisfaction relation of a fixed model need not be decidable, so standard resource-bounded complexity no longer captures the phenomenon adequately.

This paper isolates a robust framework in which infinite Kripke models still admit a finitary description amenable to classical computability theory. We work over a fixed countable set Prop of propositional variables and the basic modal language $\mathscr{L}_{\mathsf{ML}}$. We fix a computable Gödel coding $\ulcorner\cdot\urcorner$ of formulas and a computable pairing $\langle\cdot,\cdot\rangle$ on ω. We consider *computable* **S5***-models* $\mathscr{M} = (W,R,v)$, where $W \subseteq \omega$ is decidable, R is a computable equivalence relation on W, and v is computable; and we study the *global model-checking set*

$$\mathrm{MC}^{\infty}_{\mathscr{M}} := \{\langle \ulcorner\varphi\urcorner, w\rangle : \mathscr{M}, w \models \varphi\} \subseteq \omega$$

as a set of natural numbers.

The natural complexity yardsticks here are the arithmetical hierarchy and, more finely, the Ershov hierarchy inside Δ^0_2 [4]; we briefly recall the latter (see [3] for details). By a *computable approximation* of a set $A \subseteq \omega$ we mean a total computable function $\omega \times \omega \to \{0,1\}$, written $(x,s) \mapsto A_s(x)$, such that $A(x) = \lim_{s\to\infty} A_s(x)$ for every x; the parameter $s \in \omega$ is called a *stage*. A set $A \subseteq \omega$ is *n-c.e.* ($A \in \Sigma^{-1}_n$) if it admits a computable approximation that, on each input, switches value at most n times; it is *ω-c.e.*($A \in \Delta^{-1}_\omega$) if the number of switches is bounded by a computable function of the input. We write Π^{-1}_n for the complements of Σ^{-1}_n sets and $\Delta^{-1}_n := \Sigma^{-1}_n \cap \Pi^{-1}_n$ for the *balanced* level. Thus Σ^{-1}_1 is the class of c.e. sets, Δ^{-1}_1 the computable sets, and the inclusions $\Delta^{-1}_n \subsetneq \Delta^{-1}_{n+1} \subsetneq \Delta^{-1}_\omega \subsetneq \Delta^0_2$ are all strict.

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The choice of **S5** is not incidental. Modal quantification from a world w ranges exactly over its R-equivalence class $[w]$; this strong confinement is one of the characteristic semantic features of **S5** [5] and, in our computability-theoretic setting, induces a synchronisation phenomenon that prevents unbounded witness hopping across an equivalence class. Our results show that this yields a syntax-controlled bound on the dynamics of effective approximations to satisfaction, while still allowing every finite Ershov level to be realised.

2 Uniform upper bound

Recall the standard **S5**-validity of modal disjunctive normal form: every φ is effectively **S5**-equivalent to

$$\mathrm{mdnf}(\varphi) \;=\; \bigvee_{i<k}\Big(\pi_i \wedge \bigwedge_{j<m_i} \Diamond\alpha_{i,j} \wedge \bigwedge_{\ell<n_i} \Box\beta_{i,\ell}\Big),$$

with each $\pi_i, \alpha_{i,j}, \beta_{i,\ell}$ purely propositional (or empty) [5, Ch. 4]. Let $b(\varphi)$ be the number of modal-literal occurrences in $\mathrm{mdnf}(\varphi)$. Then the map $\ulcorner\varphi\urcorner \mapsto b(\varphi)$ is total computable.

For a computable **S5**-model $\mathscr{M}$, a world w, and a stage s, define the bounded stagewise valuation $\mathsf{val}_s(\mathscr{M}, w, \cdot)$ on modal literals with propositional scope by

$$\mathsf{val}_s(\mathscr{M}, w, \Diamond\gamma) = 1 \iff \exists u \le s\,\big(u \in [w] \wedge \mathscr{M}, u \models \gamma\big),$$

dually for $\Box\gamma$, and extended to Boolean combinations componentwise. For an arbitrary φ, set $\mathsf{val}_s(\mathscr{M}, w, \varphi) := \mathsf{val}_s(\mathscr{M}, w, \mathrm{mdnf}(\varphi))$.

Theorem 1. *Let $\mathscr{M}$ be a computable* **S5***-model. The map*

$$F\big(\langle\ulcorner\varphi\urcorner, w\rangle, s\big) \;:=\; \mathsf{val}_s(\mathscr{M}, w, \varphi)$$

is a total computable approximation to $\chi_{\mathrm{MC}^\infty_{\mathscr{M}}}$ such that for every input $\langle\ulcorner\varphi\urcorner, w\rangle$ the sequence $s \mapsto F(\langle\ulcorner\varphi\urcorner, w\rangle, s)$ has at most $b(\varphi)$ value changes. In particular, $\mathrm{MC}^\infty_{\mathscr{M}}$ is ω-c.e. via the explicit bounding function $\langle\ulcorner\varphi\urcorner, w\rangle \mapsto b(\varphi)$.

Proof idea. In the MDNF, every modal literal has propositional scope, so the predicate $\exists u \le s\,(u \in [w] \wedge \mathscr{M}, u \models \gamma)$ is monotone increasing in s and can switch value at most once; dually, each $\Box$-literal can switch at most once in the opposite direction. Each value change of F at $\langle\ulcorner\varphi\urcorner, w\rangle$ is therefore charged to a modal literal of $\mathrm{mdnf}(\varphi)$ flipping for the first and only time. Crucially, the bound $b(\varphi)$ depends only on the syntax of φ, uniformly in $\mathscr{M}$ and w. Correctness follows because bounded search through $[w]$ converges to full search over the equivalence class, and $\mathrm{mdnf}(\varphi)$ is effectively **S5**-equivalent to φ. □

3 Exact Ershov classification

Before turning to lower bounds, we record a structural feature of $\mathrm{MC}^\infty_{\mathscr{M}}$ that explains why exact classifications naturally land in the balanced Ershov levels.

Lemma 2. *Let* $\mathrm{tog}(\varphi) = \psi$ *if* $\varphi = \neg\psi$, *and* $\mathrm{tog}(\varphi) = \neg\varphi$ *otherwise. The map* $g(\langle\ulcorner\varphi\urcorner, w\rangle) := \langle\ulcorner\mathrm{tog}(\varphi)\urcorner, w\rangle$ *is a computable involution with*

$$x \in \mathrm{MC}^\infty_{\mathscr{M}} \iff g(x) \notin \mathrm{MC}^\infty_{\mathscr{M}}.$$

Hence $\mathrm{MC}^\infty_{\mathscr{M}} \equiv_1 \overline{\mathrm{MC}^\infty_{\mathscr{M}}}$.

Since Σ_n^{-1} and Π_n^{-1} are closed under computable preimages, Lemma 2 forces any one-sided classification to collapse to the balanced level: $\mathrm{MC}^\infty_{\mathscr{M}} \in \Sigma_n^{-1}$ implies $\mathrm{MC}^\infty_{\mathscr{M}} \in \Delta_n^{-1}$. Thus the natural targets for exact classification are the balanced classes.

Theorem 3. *For every $n \geq 1$ there exists a computable* **S5***-model $\mathscr{M}_n$ such that $\mathrm{MC}^\infty_{\mathscr{M}_n} \in \Delta_{n+1}^{-1} \setminus \Delta_n^{-1}$.*

Proof idea. Fix an effective listing $(E_e^n)_{e\in\omega}$ of the n-c.e. sets, each with a stagewise approximation $E_e^n[s](x) \in \{0,1\}$ such that $E_e^n[0](x) = 0$ and $\lim_{s\to\infty} E_e^n[s](x) = E_e^n(x)$ flipping at most n times. Let

$$c_e(x,s) := \ |\{t \leq s : E_e^n[t](x) \neq E_e^n[t-1](x)\}|$$

count the flips up to stage s.

The frame is the same in all $\mathscr{M}_n$: $W := \omega \times \omega$, with $w_{e,s} R w_{e',t} \iff e = e'$, so the equivalence classes are the columns $\mathscr{C}_e = \{w_{e,s} : s \in \omega\}$. Inside column e, propositional variables track which threshold has been crossed by the local counter $c_e(x_e, s)$, where $x_e := \langle \ulcorner \varphi_n \urcorner, \langle e, 0\rangle\rangle$ is the diagonal input. The fixed test formula φ_n is a parity test over the limit value $C_e := \lim_s c_e(x_e, s) \leq n$:

$$\varphi_{2m+1} \ := \ \Box q_1 \ \vee \ \bigvee_{k=1}^{m} \big(\Diamond r_{2k} \wedge \Box q_{2k+1}\big),$$

$$\varphi_{2m} \ := \ \Box q_1 \ \vee \ \bigvee_{k=1}^{m-1} \big(\Diamond r_{2k} \wedge \Box q_{2k+1}\big) \ \vee \ \Diamond r_{2m},$$

designed so that

$$\mathscr{M}_n, w_{e,0} \models \varphi_n \iff C_e \text{ is even} \iff E_e^n(x_e) = 0. \tag{1}$$

This is the diagonal step: if $\mathrm{MC}^\infty_{\mathscr{M}_n} \in \Delta_n^{-1} \subseteq \Sigma_n^{-1}$, then $E_e^n = \mathrm{MC}^\infty_{\mathscr{M}_n}$ for some e, and evaluating both sides on x_e via (1) yields $E_e^n(x_e) = 1 \iff E_e^n(x_e) = 0$, a contradiction. For the upper bound, each column passes through at most $n+1$ semantic phases $0, 1, \ldots, C_e$; truth in $\mathscr{M}_n$ at $w_{e,t}$ agrees with truth in a finite **S5**-model with one world per phase, and tracking the current phase yields a computable approximation with at most $n+1$ value changes, hence $\mathrm{MC}^\infty_{\mathscr{M}_n} \in \Sigma_{n+1}^{-1}$. Combined with Lemma 2, this gives $\mathrm{MC}^\infty_{\mathscr{M}_n} \in \Delta_{n+1}^{-1}$. □

By uniformising the construction over all n, one also obtains a computable **S5**-model $\mathscr{M}_\omega$ such that $\mathrm{MC}^\infty_{\mathscr{M}_\omega}$ is properly ω-c.e. Together with trivial examples at level Δ_1^{-1} (e.g., finite models), Theorems 1 and 3 yield a complete classification: the family $\{\mathrm{MC}^\infty_{\mathscr{M}} : \mathscr{M} \text{ computable } \mathbf{S5}\text{-model}\}$ realises exactly the balanced finite Ershov levels Δ_n^{-1} ($n \geq 1$) and Δ_ω^{-1}.

The main point is that **S5** remains tame over infinite computable models: global model checking is always ω-c.e., yet every finite balanced Ershov level is realised. We expect global model checking to escape Δ_ω^{-1} in weaker systems such as **S4**, **K**, or **Grz** where equivalence-class confinement (which underlies our syntax-driven bound) is lost.

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Interpolation Above S4

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We complete Maksimova's classification of the normal extensions of **S4** with interpolation. In particular, we prove Craig interpolation for the six extensions of **S4** for which Craig interpolation was still open. The proof strategy builds upon the ideas of Smoryński, but employs a novel approach using Fine's frame formulas for splitting clusters.

By a celebrated result of Maksimova [6], exactly eight super-intuitionistic logics have the Craig interpolation property (CIP). Shortly thereafter, Maksimova showed, using the intimate connection between super-intuitionistic logics and extensions of the modal logic **S4**, that at most 38 normal extensions of **S4** have the CIP [7], and reduced this to 37 in [9]. Moreover, in [8], she proved the CIP for the majority of these logics, more specifically, the ones that are canonical. Subsequently, Boolos [1], Rautenberg [12], and Maksimova [10] proved the CIP for, in total, another four of the logics, leaving only eight cases open. Two of these are claimed to have the CIP in [11]. For the final six logics, the status of the CIP has remained open. Regarding the deductive interpolation property (DIP), the story is similar, with at most 49 normal extensions of **S4** enjoying the DIP, and six open cases coinciding exactly with those for the CIP. A complete overview of the situation is presented in [5].

We prove the CIP for the remaining normal extensions of **S4**. Since for modal logics, the CIP implies the DIP, this completes Maksimova's characterization for both the CIP and the DIP in normal extensions of **S4**. This resolves Problem 14.3 in Chagrov and Zakharyashev's standard reference on modal logic [2], which was also posed recently in [4, Open Problem 9]. The full paper is available on arXiv [13].

A logic Λ is said to have the *Craig interpolation property (CIP)* if for all formulas φ, ψ with $\varphi \to \psi \in \Lambda$, there exists a formula χ whose atomic propositions appear in both φ and ψ, such that $\varphi \to \chi, \chi \to \psi \in \Lambda$. In this case, the formula χ is called an *interpolant* for $\varphi \to \psi$.

Maksimova [7] showed that every normal extension of **S4** with the CIP, is of a certain shape: its intuitionistic fragment Λ has the CIP and it is complete with respect to finite $\tau(\Lambda)$-frames with a bound on the size of their clusters, where $\tau(\Lambda)$ is the smallest modal companion of Λ.

More precisely, for a super-intuitionistic logic Λ the modal logic $\Gamma(\Lambda, m, n)$ is the normal extension of **S4** complete with respect to the finite $\tau(\Lambda)$-frames such that every final cluster has size at most m and every non-final cluster has size at most n [5, pag.73].

Every normal extension of **S4** with the CIP is of the form $\Gamma(\Lambda, m, n)$, for some super-intuitionistic logic Λ with the CIP and $m, n \in \{1, 2, \omega\}$. For example, we have $\Gamma(\mathbf{Int}, \omega, \omega) = \mathbf{S4}$, $\Gamma(\mathbf{Int}, 1, 1) = \mathbf{Grz}$, $\Gamma(\mathbf{KC}, \omega, \omega) = \mathbf{S4.2}$, and $\Gamma(\mathbf{Cl}, \omega, 0) = \mathbf{S5}$, where **Int** denotes intuitionistic propositional logic, **Cl** denotes classical propositional logic, and $\mathbf{KC} = \mathbf{Int} + (\neg p \vee \neg\neg p)$. The six normal extensions of **S4** for which the CIP remained open, are

$$\Gamma(\Lambda, m, 2) \qquad \text{for } \Lambda \in \{\mathbf{Int}, \mathbf{KC}\} \text{ and } m \in \{1, 2, \omega\},$$

see [5, pp. 256–257]. Additionally, for $\Gamma(\Lambda,2,1)$ with $\Lambda \in \{\mathbf{Int},\mathbf{KC}\}$, the CIP was claimed in [11, p. 462]. We show that these eight logics have the CIP; in fact, we give a uniform proof of the CIP for the logics $\Gamma(\Lambda,m,n)$ with $\Lambda \in \{\mathbf{Int},\mathbf{KC}\}$ and $m,n \in \{1,2,\omega\}$.

Our approach is based on the method that Smoryński [14] used to show the CIP for **GL**. This approach was also used by Boolos [1] to prove the CIP for **Grz** and by Maksimova [10] for two further normal extensions of **S4**. Smoryński's method relies on syntactically constructing finite models. First, it is assumed that an implication does not have an interpolant. Using this, a finite countermodel for the implication is constructed. For logics that impose restrictions on the clusters in their frames, it is usually necessary to tweak the definition of the accessibility relation, so that the constructed model is a model of the logic, cf. [1, 10, 14]. Thus, every implication lacking an interpolant is invalid in the logic, and the CIP is obtained.

The key difference of our approach compared to the previous ones is that we employ a more flexible and semantic approach towards abiding by the restrictions on clusters. We first construct a standard Smoryński model $\mathfrak{M}$, which is only a model for **S4** (or **S4.2**). The cluster restrictions are then expressed using Fine's frame formulas [3], and it follows from the Truth Lemma for Smoryński models that $\mathfrak{M}$ validates (sufficiently large) substitutions of these frame formulas. Using these, we show that every cluster contains a small enough *adequate* subset of points, which together do not rely on other points in the cluster for witnessing the satisfaction of formulas. This then finally allows us to modify the accessibility relation into one that does not contain clusters that are too large, while preserving the Truth Lemma.

Therefore, we are able to show for the logics of interest that if an implication does not have an interpolant, then it is not valid, yielding our main result.

Theorem 1. *Let* $\Lambda \in \{\mathbf{Int},\mathbf{KC}\}$ *and* $m,n \in \{1,2,\omega\}$. *Then the logic* $\Gamma = \Gamma(\Lambda,m,n)$ *has the Craig interpolation property.*

Our Main Theorem completes Maksimova's classification of the normal extensions of **S4** with the CIP [5, Theorem 8.45, pp. 256–257]. Moreover, for modal logics, the CIP implies the DIP, and thus, this completes the classification for the DIP as well. Together, this resolves Problem 14.3 of Chagrov and Zakharyaschev [2, pag. 469]. We obtain the following explicit characterization, where we recall that the super-intuitionistic logics with the CIP are exactly those listed in table 2 [6].

Corollary 2. *There are exactly 37 normal extensions of* **S4** *with the Craig interpolation property and exactly 49 with the deductive interpolation property. In particular:*

1. *A normal extension of* **S4** *has the CIP if and only if it is one of the following logics:*

Fm (*the inconsistent logic*);	
$\Gamma(\Lambda,m,n)$,	$\Lambda \in \{\mathbf{Int},\mathbf{KC}\}$, $m,n \in \{1,2,\omega\}$;
$\Gamma(\Lambda,n,1)$, $\Gamma(\Lambda,1,n)$,	$\Lambda \in \{\mathbf{LP}_2,\mathbf{LV},\mathbf{LS}\}$, $n \in \{1,2,\omega\}$;
$\Gamma(\mathbf{Cl},n,0)$,	$n \in \{1,2,\omega\}$.

2. *A normal extension of* **S4** *has the DIP if and only if it is a logic mentioned in* (i) *or it is one of the following logics:*

$$\Gamma(\Lambda,m,n), \quad \Lambda \in \{\mathbf{LP}_2,\mathbf{LV},\mathbf{LS}\},\ m,n \in \{2,\omega\}.$$

Table 2: The super-intuitionistic logics with the Craig interpolation property.

Logic	Axiomatization
Int	Intuitionistic logic
KC	$\mathbf{Int} + \neg p \vee \neg\neg p$
$\mathbf{LP}_2$	$\mathbf{Int} + p \vee (p \to (q \vee \neg q))$
LV	$\mathbf{LP}_2 + (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$
LS	$\mathbf{LP}_2 + \neg p \vee \neg\neg p$
LC	$\mathbf{Int} + (p \to q) \vee (q \to p)$
Cl	$\mathbf{Int} + p \vee \neg p$
Fm	$\mathbf{Int} + \bot$

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A New Positive Free Modal Logic with Definite Descriptions

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Introduction Definite descriptions are usually formalized as the expressions of the form $\iota x\varphi$, where x is an individual variable and φ is a formula. The earliest and most influential theories of definite descriptions are due to Frege [3] and Russell [9]. In this abstract, we focus on more recent theories, namely those of Garson [4] and Kürbis [7]. Both are based on positive free logic, while Garson's theory is specifically based on positive free *modal* logic. Garson [4] originally presented his logics semantically and by means of natural deduction systems; later on, Indrzejczak [5] provided cut-free sequent calculi for them. A desirable property of these logics is that two improper definite descriptions, which fail to uniquely denote objects, can denote different objects. The problem, however, is that two intuitively valid formulas $a = b \to \iota xP(xa) = \iota xP(xb)$ and $\iota xP(xx) = \iota yP(yy)$ are not valid in the semantics characterizing his logics.

To overcome this problem, we develop a new positive free **S5**-based modal logic with definite descriptions $\mathbf{S5_{dd}}$. In our definition of the semantic interpretation of $\iota x\varphi$, we obviously refer to unique objects. However, following Kürbis [7], our second source of inspiration, it is not necessary for the objects to exist. While quantifiers in our logic have the so-called existential support (which is typical for free logics), definite descriptions may refer to any element of the domain, including non-denoting ones. In the case of improper definite descriptions, our semantics may at first glance look Fregean, since it employs a special designated object, **null**. However, in our setting most improper descriptions (like "the Land of the Persian Gulf") need not denote **null**. They denote the unique objects that might not exist in the world under consideration. The object **null** is reserved for those descriptions whose impropriety is forced by the theory across all worlds (like "the land that is not identical to itself"). We adopted **S5** modality just because the resulting semantics becomes simple. It would be of course philosophically interesting to add a different kind of modality to the modality-free fragment of $\mathbf{S5_{dd}}$, but we do not attempt it in this abstract. We also present a sound and complete cut-free hypersequent calculus for $\mathbf{S5_{dd}}$. It is well known that while **S5** has no cut-free ordinary sequent calculus, it admits a number of cut-free hypersequent calculi; for a survey, see [2]. The cut elimination theorem is proved in a semantic way.

Positive Free Modal Logic $\mathbf{S5_{dd}}$ Let FV and BV be countably infinite disjoint sets of *free variables* $a, b, c, \ldots$ and *bound variables* $x, y, z, \ldots$. The language $\mathscr{L}$ consists of the union $\mathsf{Var} = \mathsf{FV} \cup \mathsf{BV}$, the set $\{\mathsf{null}\}$ of a *constant* null, the union $\mathsf{Pred} = \bigcup_{n\in\mathbb{N}} \mathsf{Pred}^n$ of countable sets Pred^n of *n-ary predicate symbols* $P, Q, \ldots$ and the set of logical constants $=, E, \neg, \to, \forall, \Box, \iota$ where E is the existential predicate.

The members of Var are called just *variables*. The set Term of *semi-terms* and the set Form of *semi-formulas* in $\mathscr{L}$ are simultaneously defined as follows.

$$\mathsf{Term} \ni t ::= a \mid x \mid \mathsf{null} \mid \iota x\varphi$$
$$\mathsf{Form} \ni \varphi ::= P(t_1 \cdots t_n) \mid t = s \mid E(t) \mid \neg\varphi \mid (\varphi \to \varphi) \mid \forall x\varphi \mid \Box\varphi$$

Semi-formulas of the form $P(t_1 \cdots t_n)$, $t = s$ and $E(t)$ are called *atomic*, and semi-terms of the form $\iota x\varphi$ are called *definite descriptions*. A *term* and a *formula* are respectively a semi-term and a semi-formula in which no bound variables freely occur.

Definition 1. A *model* is a tuple $\langle W, D, \mathbf{null}, I\rangle$, where W is a non-empty set of worlds $w, v, \ldots$; D is a function mapping each $w \in W$ to a possibly empty set D_w of *objects* $o, o', \ldots$ in w; **null** is some distinguished object not in $D_W := \bigcup_{w\in W} D_w \neq \varnothing$ and $D_W^+ := D_W \cup \{\mathbf{null}\}$; I is a function mapping each $w \in W$ to an interpretation function I_w that maps null to **null** and each n-ary predicate symbol P to a subset $I_w(P)$ of ${D_W^+}^n$. An *assignment* σ is a function from Var to D_W, and $\sigma\binom{r}{o}$ is the assignment obtained from σ by mapping $r \in \mathsf{Var}$ to $o \in D_W$. We note that $\sigma(r) \neq \mathbf{null}$ for all $r \in \mathsf{Var}$.

Definition 2. Let M be a model $\langle W, D, \mathbf{null}, I\rangle$, w a world, σ an assignment and φ a semi-formula. The *extended assignment* $\sigma_w \colon \mathsf{Term} \to D_W^+$ and the *satisfaction relation* $M, \sigma, w \models \varphi$ are recursively defined as follows, where atomic cases and propositional connective cases are supposed to be defined as usual.

$$\begin{array}{ll}
\sigma_w(r) = \sigma(r), \quad \sigma_w(\mathsf{null}) = I_w(\mathsf{null}) & \\
\sigma_w(\iota x\varphi) = o & \text{if there exists some unique } o \in D_W \text{ such that } M, \sigma\binom{x}{o}, w \models \varphi; \\
\sigma_w(\iota x\varphi) = \mathbf{null} & \text{if such a unique object does not exist} \\
\quad M, \sigma, w \models \forall x\varphi & \text{iff} \quad \text{for all } o \in D_w,\ M, \sigma\binom{x}{o}, w \models \varphi \\
\quad M, \sigma, w \models \Box\varphi & \text{iff} \quad \text{for all } v \in W,\ M, \sigma, v \models \varphi
\end{array}$$

The validity of formulas are defined as usual and denoted by $\models \varphi$. We refer to the set of valid formulas as *the positive free modal logic with definite descriptions* $\mathbf{S5_{dd}}$.

Let us define the substitutions s_t^z and φ_t^z of a term t for a variable z in a semi-term s and a semi-formula φ as usual, and say that a bound variable x is in the modal contexts of a semi-formula φ (or a semi-term s) if there exists some semi-formula $\Box\psi$ occurring in φ (or s) such that x freely occurs in ψ. Here is a list of remarkable (in)valid formulas in this semantics.

Proposition 3. *Let a, b be free variables, x, y bound variables, P a binary predicate symbol.*

(1) $\models t = s \to (\varphi_t^y \to \varphi_s^y)$
where y is not in the modal contexts of φ or $t, s \in \mathsf{FV} \cup \{\mathsf{null}\}$

(2) $\models \neg E(\mathsf{null})$

(3) $\models \forall y(y = \iota x\varphi \to \varphi_y^x \wedge \forall x(\varphi \to x = y))$

(4) $\models \forall y(\varphi_y^x \wedge \forall x(\varphi \to x = y) \to y = \iota x(E(x) \wedge \varphi))$

(5) $\models a = b \to \iota xP(xa) = \iota xP(xb)$

(6) $\models \iota xP(xx) = \iota yP(yy)$

(7) $\not\models \neg E(\iota x\varphi) \wedge \neg E(\iota y\psi) \to \iota x\varphi = \iota y\psi$

$$\frac{}{\varphi \Rightarrow \varphi}\,(Ax) \qquad \frac{}{E(\mathsf{null}) \Rightarrow}\,(E_{\mathsf{null}}) \qquad \frac{}{a = \mathsf{null} \Rightarrow}\,(=_{\mathsf{null}})$$

$$\frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H}\,(IW) \qquad \frac{H}{G \mid H}\,(EW) \qquad \frac{\Gamma \Rightarrow \Delta \mid \mu, \Pi \Rightarrow \Sigma \mid H}{\mu, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H}\,(TS)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \mid H \qquad \varphi, \Pi \Rightarrow \Sigma \mid G}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}\,(Cut) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg\varphi \mid H}\,(\Rightarrow \neg) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \mid H}{\neg\varphi, \Gamma \Rightarrow \Delta \mid H}\,(\neg \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi \mid H}{\Gamma \Rightarrow \Delta, \varphi \to \psi \mid H}\,(\Rightarrow \to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \mid H \qquad \psi, \Pi \Rightarrow \Sigma \mid G}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}\,(\to \Rightarrow) \qquad \frac{E(c), \Gamma \Rightarrow \Delta, \varphi^x_c \mid H}{\Gamma \Rightarrow \Delta, \forall x\varphi \mid H}\,(\Rightarrow \forall)$$

$$\frac{E(a), \varphi^x_a, \Gamma \Rightarrow \Delta \mid H}{E(a), \forall x\varphi, \Gamma \Rightarrow \Delta \mid H}\,(\forall \Rightarrow) \qquad \frac{\Rightarrow \varphi \mid H}{\Rightarrow \Box\varphi \mid H}\,(\Rightarrow \Box) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Box\varphi, \Gamma \Rightarrow \Delta \mid H}\,(\Box \Rightarrow)$$

$$\frac{E(a) \Rightarrow \mid H}{H}\,(E) \qquad \frac{a = c, \Gamma \Rightarrow \Delta, \varphi^x_c \mid H \qquad \varphi^x_c, \Pi \Rightarrow \Sigma, a = c \mid G}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, a = \iota x\varphi \mid H \mid G}\,(\Rightarrow \iota_1)$$

$$\frac{r = r, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid H}\,(Id) \qquad \frac{\Gamma \Rightarrow \Delta, a = b, \varphi^x_b \mid H \qquad a = b, \varphi^x_b, \Pi \Rightarrow \Sigma \mid G}{a = \iota x\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}\,(\iota \Rightarrow 1)$$

$$\frac{\alpha^y_t, \Gamma \Rightarrow \Delta \mid H}{t = s, \alpha^y_s, \Gamma \Rightarrow \Delta \mid H}\,(Sub_1) \qquad \frac{c = \iota x\varphi, \Gamma \Rightarrow \Delta, \alpha^y_c \mid H \qquad \Pi \Rightarrow \Sigma, b = \iota x\varphi, \alpha^y_{\mathsf{null}} \mid G}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \alpha^y_{\iota x\varphi} \mid H \mid G}\,(\Rightarrow \iota_2)$$

$$\frac{\alpha^y_s, \Gamma \Rightarrow \Delta \mid H}{t = s, \alpha^y_t, \Gamma \Rightarrow \Delta \mid H}\,(Sub_2) \qquad \frac{c = \iota x\varphi, \alpha^y_c, \Gamma \Rightarrow \Delta \mid H \qquad \alpha^y_{\mathsf{null}}, \Pi \Rightarrow \Sigma, b = \iota x\varphi \mid G}{\alpha^y_{\iota x\varphi}, \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}\,(\iota \Rightarrow 2)$$

where, in any presented rule, $\mu \equiv \Box\varphi$ for some formula φ or $\mu \equiv r_1 = r_2$ for some $r_1, r_2 \in \mathsf{FV} \cup \{\mathsf{null}\}$; c is fresh in $a, \alpha, \varphi, \Gamma, \Delta, \Pi, \Sigma, H, G$; $r \in \mathsf{FV} \cup \{\mathsf{null}\}$; α is an atomic semi-formula such that y is not in the modal contexts of α. Moreover, in $(\Rightarrow \iota_2)$ and $(\iota \Rightarrow 2)$, y freely occurs in α and $\alpha \not\equiv b = y$ for any $b \in \mathsf{FV}$.

Table 3: Hypersequent calculus $\mathsf{GS5}_{\mathbf{dd}}$

Hypersequent calculus $\mathsf{GS5}_{\mathbf{dd}}$ A *sequent* is a pair of finite sets Γ and Δ of formulas, denoted by $\Gamma \Rightarrow \Delta$. A *hypersequent* is a finite set of sequents $S_1, \ldots, S_n$, denoted by $S_1 \mid \cdots \mid S_n$. A *hypersequent calculus* $\mathsf{GS5}_{\mathbf{dd}}$ for $\mathbf{S5}_{\mathbf{dd}}$ is given by Table 1, where H, G are hypersequents and $S_1 \mid \cdots \mid S_n \mid H \mid G$ denotes the hypersequent $\{S_1, \ldots, S_n\} \cup H \cup G$. The derivability and the semantic consequence of a hypersequent H from a set of hypersequents $\mathscr{H}$ are defined as in [1], respectively denoted by $\mathscr{H} \vdash_{\mathsf{GS5}_{\mathbf{dd}}} H$ and $\mathscr{H} \models H$. We also write $\mathsf{GS5}^{-}_{\mathbf{dd}}$ to mean the (Cut)-free fragment of $\mathsf{GS5}_{\mathbf{dd}}$.

Some rules in our hypersequent calculus are similar to but well-behaved than those of the existing hypersequent calculi, and some are completely new. For the lack of space, here we illustrate only (TS). The rule (TS) is a generalization of the rule $(\mid TR \Rightarrow)$ adopted in [6, 281], and the semantic intuition behind it is that the truth of modal formulas and equalities on rigid terms do not change between worlds. This rule makes Restall's [8] left $\Box$ rule derivable:

$$\frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Box\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}.$$

Thus the set of $(TS), (\Box \Rightarrow), (\Rightarrow \Box)$ can successfully capture the behavior of **S5** modality. Furthermore, it follows from the (strong) soundness and the completeness claimed below that the following rules

$$\frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \mu \mid H}{\Gamma \Rightarrow \Delta, \mu \mid \Pi \Rightarrow \Sigma \mid H} \qquad \frac{\Gamma \Rightarrow \Delta, a = b \mid H}{\Gamma \Rightarrow \Delta, \Box a = b \mid H}\,(\Rightarrow = \Box) \qquad \frac{a = b, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \Box a \neq b \mid H}\,(\Rightarrow \neq \Box),$$

whose restricted forms are adopted as rules in [6, 290], are cut-free admissible.[1]

The strong soundness and the completeness are proved as in [1].

Theorem 4 (Strong soundness of $\mathsf{GS5}_{\mathbf{dd}}$ and completeness of $\mathsf{GS5}^{-}_{\mathbf{dd}}$)**.** *Let $\mathscr{H} \cup \{H\}$ be a set of hypersequents. Then $\mathscr{H} \vdash_{\mathsf{GS5}_{\mathbf{dd}}} H$ implies $\mathscr{H} \models H$, and $\models H$ implies $\vdash_{\mathsf{GS5}^{-}_{\mathbf{dd}}} H$.*

Corollary 5 (Cut elimination)**.** *Let H be a hypersequent. If $\vdash_{\mathsf{GS5}_{\mathbf{dd}}} H$ then $\vdash_{\mathsf{GS5}^{-}_{\mathbf{dd}}} H$.*

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[1] In [6], the application of (TS) is restricted to modal formulas of the form $\Box\varphi$ and $\Diamond\varphi$. As a result, $(\Rightarrow = \Box)$ and $(\Rightarrow \neq \Box)$ are added for ensuring the derivability of $a = b \Rightarrow \Box a = b$ and $a \neq b \Rightarrow \Box a \neq b$. However, the addition of these rules is still weak for the cut elimination since $a = b, a' = b' \Rightarrow \Box(a = b \wedge a' = b')$ is not cut-free derivable in the calculus presented there. The formulation of (TS) is inspired by personal discussions between the first author and Indrzejczak in order to solve this problem.

Model Checking for Distributed Knowing How

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We explore the complexity of the model checking problem for distributed knowing how. We obtain a Δ_2^p upper bound and a DP-hard lower bound for the problem.

1 Introduction

Knowing-how logics study agents' abilities to achieve goals by suitable actions. This makes them particularly relevant to artificial intelligence, especially to planning, where the main question is whether a desired objective can be guaranteed by some plan. The logic of distributed knowing how [1] generalizes two existing frameworks of knowing how. The planning-based framework, initiated in [5, 6], treats knowing how as the existence of a multi-step plan that can be executed, always terminates, and guarantees the goal, while the coalition-based framework, initiated in [2, 3], treats group know-how as the existence of a one-step joint action that ensures the goal.

In this work, we give some preliminary complexity bounds for the model checking problem for the logic of distributed knowing how. More precisely, we introduce an oracle-based model checking algorithm in Δ_2^p, the class of problems decidable in polynomial time with polynomially many calls to an NP oracle. The main difficulty lies in the Kh_G-modality. Instead of explicitly constructing the distributed action set A_G^*, our procedure first computes the satisfaction set of the inner formula and then runs a fixpoint algorithm over the quotient set $S/\sim_G$. We also show that the problem is already DP-hard, by a reduction from SAT-UNSAT [4].

2 Basic Definitions

Let $\mathscr{P}$ be a denumerable set of proposition symbols and let I be a finite set of agents. The language DKH is defined by the grammar

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_G\varphi \mid Kh_G\varphi$$

where $p \in \mathscr{P}$ and $G \subseteq I$. A model $\mathscr{M}$ is a tuple $\left\langle S, \{\sim_i\}_{i\in I}, \{A_G\}_{G\subseteq I}, \left\{ \xrightarrow{a} \mid a \in \bigcup_{G\subseteq I} A_G \right\}, V \right\rangle$, where S is a set of states; $\sim_i$ is an equivalence relation on S for each $i \in I$; A_G is a set of atomic group actions for each $G \subseteq I$ such that: $G \subsetneq H$ implies $A_G \cap A_H = \varnothing$; $A_\varnothing = \varnothing$; $\xrightarrow{a}$ is a binary relation on S for each $a \in \bigcup_{G\subseteq I} A_G$, called the transition relation of a; $V : \mathscr{P} \to 2^S$ is a valuation function. The family $\{A_G\}_{G\subseteq I}$ is part of the model, so the input size depends on how this family is represented. We assume an explicit representation in which only non-empty A_G's are listed.

For each group $G \subseteq I$, the distributed indistinguishability relation is defined by $\sim_G := \bigcap_{i\in G} \sim_i$. We write $[s]_G$ for the $\sim_G$-equivalence class of s, and $[S]_G$ for the corresponding quotient set. Distributed

actions are defined inductively as in [1]: $A^*_\varnothing = \varnothing$, $A^*_i = A_i$, and for larger groups G, A^*_G additionally contains tuples $\langle d_0, \ldots, d_n \rangle$ obtained from pairwise disjoint subgroups forming a non-trivial partial partition of G, where each $d_k \in A^*_{G_k}$. For each distributed action $d = \langle d_0, \ldots, d_n \rangle \in A^*_I$, the distributed transition relation is defined by $\xrightarrow{d} := \bigcap_{0 \le k \le n} \xrightarrow{d_k}$.

A distributed action $d \in A^*_I$ is executable on a nonempty set $X \subseteq S$ if every state in X admits a d-successor. A strategy for group G is a partial function $\sigma_G : [S]_G \to A^*_G$ assigning to each equivalence class an executable distributed action; in particular, the empty function is always a strategy. A possible execution of σ_G is a finite or infinite sequence $[s_0]_G[s_1]_G \cdots$ such that $[s_j]_G \xrightarrow{\sigma_G([s_j]_G)} [s_{j+1}]_G$ whenever $\sigma_G([s_j]_G)$ is defined. A complete execution is either infinite or ends at a class outside $\mathrm{dom}(\sigma_G)$. We write $\mathrm{CELeaf}(\sigma_G, s)$ and $\mathrm{CEInner}(\sigma_G, s)$ for the sets of leaf-nodes and inner-nodes, respectively, of all complete executions starting from $[s]_G$.

The semantics is defined as follows. Given a model $\mathscr{M}$, a state $s \in S$, and a formula $\varphi \in \mathsf{DKH}$, we omit the Boolean cases.

$\mathscr{M}, s \Vdash K_G\varphi$	*iff*	$\mathscr{M}, s' \Vdash \varphi$ for all $s' \in [s]_G$
$\mathscr{M}, s \Vdash Kh_G\varphi$	*iff*	there is a strategy σ_G of G such that: (1) $[t]_G \subseteq [\![\varphi]\!]^{\mathscr{M}}$ for all $[t]_G \in \mathrm{CELeaf}(\sigma_G, s)$, and (2) all complete executions of σ_G starting from $[s]_G$ are finite.

3 Model Checking

The model checking procedure $\textsc{CheckerDKH}(\mathscr{M}, s, \varphi)$ decides whether the input state s satisfies the formula φ on a finite model $\mathscr{M}$. It is defined via an evaluation procedure $\textsc{Eval}(\mathscr{M}, \varphi)$, which computes the satisfaction set of φ bottom-up, following the structure of formulas. Thus, $\textsc{CheckerDKH}(\mathscr{M}, s, \varphi)$ returns true iff $s \in \textsc{Eval}(\mathscr{M}, \varphi)$. The Boolean cases and the K-modality are treated as usual. The non-trivial case is the evaluation of a Kh_G-formula. For such a formula $Kh_G\psi$, Algorithm 1 first computes $\textsc{Eval}(\mathscr{M}, \psi)$ and then performs a monotone fixpoint construction over the quotient set $S/{\sim_G}$. Starting from the $\sim_G$-equivalence classes already contained in the target set, the algorithm repeatedly adds a class E whenever there exists a distributed action that is executable on E and whose possible successor classes are all contained in the current approximation.

The existence of such a witness is checked by an NP oracle. Algorithm 2 gives the corresponding nondeterministic procedure. Note that an actual distributed action can be nested, but that it can always be flattened via Proposition 14 in [1]. In particular, IsValidDAction checks whether the guessed tuple can be viewed as an element of A^*_G, by assigning each action to a suitable subgroup and verifying that the chosen subgroups are defined and pairwise disjoint. Once the fixpoint is reached, EvalKH returns the union of all states belonging to the winning equivalence classes. By construction, a class is added at some iteration iff it admits a distributed action whose possible successor classes are all winning. Hence, the resulting fixpoint coincides exactly with the set of states satisfying $Kh_G\psi$. Since the fixpoint has at most $|S/{\sim_G}|$ iterations and each iteration makes only polynomially many calls to the NP oracle HasWinDistribution, the total number of oracle calls is polynomial. All remaining operations are polynomial-time set computations. Hence, the algorithm is in Δ^p_2.

For the lower bound, we reduce from the DP-complete problem SAT-UNSAT [4]: given two CNF formulas Φ and Ψ, decide whether Φ is satisfiable and Ψ is unsatisfiable. Let $\Phi = C_1 \wedge \cdots \wedge C_m$ be over variables $x_1, \ldots, x_n$, and let $\Psi = D_1 \wedge \cdots \wedge D_r$ be over variables $y_1, \ldots, y_\ell$. We construct a DKH model checking instance $\langle \mathscr{M}, s, Kh_{G_\Phi} p \wedge \neg Kh_{G_\Psi} q \rangle$, where p and q are distinct proposition letters. The model consists of two independent copies, one for Φ and one for Ψ, sharing only the initial state s. In the Φ-

Algorithm 1: Evaluation of Kh-modality

Function EVALKH($\mathscr{M}, G, \varphi$)**:**
- $X \leftarrow$ EVAL($\mathscr{M}, \varphi$) ; $\sim_G \leftarrow \bigcap_{i \in G} \sim_i$; $\mathscr{C}_G \leftarrow S/\sim_G$; $W \leftarrow \{E \in \mathscr{C}_G \mid E \subseteq X\}$;
- *changed* $\leftarrow$ True ;
- **while** *changed* **do**
  - *changed* $\leftarrow$ False ;
  - **foreach** $E \in \mathscr{C}_G \setminus W$ **do**
    - **if** HASWINDISTRIBUTION($\mathscr{M}, G, E, W$) **then**
      - $W \leftarrow W \cup \{E\}$;
      - *changed* $\leftarrow$ True ;
- **return** $\bigcup_{E \in W} E$;

Algorithm 2: A nondeterministic procedure

Function HASWINDISTRIBUTION($\mathscr{M}, G, E, W$)**:**
- $A_G^+ \leftarrow \bigcup_{H \subseteq G} A_H$;
- **nondeterministically guess** a pair $(m, \langle a_1, \ldots, a_m \rangle)$, $1 \leq m \leq |G|, a_i \in A_G^+$;
- **if** *not* ISVALIDDACTION($\mathscr{M}, G, \langle a_1, ..., a_m \rangle$) **then**
  - **return** False ;
- $\rightarrow^d \leftarrow \bigcap_{k=1}^m \rightarrow^{a_k}$;
- **if** $\big(\forall x \in E\, \exists y \in S\, x \rightarrow^d y\big) \wedge \big(\forall x \in E\, \forall y \in S\, (x \rightarrow^d y \Rightarrow [y]_G \in W)\big)$ **then**
  - **return** True;
- **return** False ;

copy, each variable x_i is represented by an agent in $G_\Phi = \{1, \ldots, n\}$ with two fresh singleton actions t_i^Φ and f_i^Φ, representing the truth values of x_i; the Ψ-copy is defined analogously with a disjoint group G_Ψ and actions t_i^Ψ, f_i^Ψ. For the empty group, we define its action set to be empty. AOnly singleton groups are assigned atomic actions, all epistemic relations are identities, and only non-empty action sets are explicitly listed. The state space contains s, good states g_Φ and g_Ψ, and bad states b_j^Φ, b_j^Ψ for the clauses of Φ, Ψ, with $V(p) = \{g_\Phi\}$ and $V(q) = \{g_\Psi\}$. In the Φ-copy, every action t_i^Φ, f_i^Φ leads from s to g_Φ. Moreover, $s \xrightarrow{t_i^\Phi} b_j^\Phi$ iff x_i does not occur in C_j, and $s \xrightarrow{f_i^\Phi} b_j^\Phi$ iff $\neg x_i$ does not occur in C_j. The Ψ-copy is defined in the same way using g_Ψ and the clauses D_j. There are no other transitions.

Intuitively, each copy encodes a truth assignment by a distributed action choosing one action for each variable. For example, consider the Φ-copy. If Φ is satisfiable, choose the distributed action according to a satisfying assignment. For any clause C_j, some selected literal is true in C_j; by construction, the corresponding action has no transition from s to the bad state b_j^Φ. Since distributed transitions are intersections, b_j^Φ cannot be reached. This applies to every clause, so all bad states are eliminated and the only successor is g_Φ; hence $Kh_{G_\Phi} p$ holds. Conversely, if Φ is unsatisfiable, then no consistent partial assignment selected by a distributed action can hit all clauses; otherwise it could be extended to a satisfying total assignment. Hence, some clause C_j has no true selected literal. Then every selected action has a transition from s to b_j^Φ, so b_j^Φ remains reachable; since $b_j^\Phi \notin V(p)$, $Kh_{G_\Phi} p$ fails. Thus $\mathscr{M}, s \Vdash Kh_{G_\Phi} p$ iff Φ is satisfiable. Similarly, $\mathscr{M}, s \Vdash Kh_{G_\Psi} q$ iff Ψ is satisfiable. Therefore, $\mathscr{M}, s \Vdash Kh_{G_\Phi} p \wedge \neg Kh_{G_\Psi} q$ iff Φ is satisfiable and Ψ is unsatisfiable. The construction can be computed in polynomial time. Hence, model checking for the logic of distributed knowing how is DP-hard.

4 Conclusion and Future Work

We established a Δ_2^p upper bound and a DP-hard lower bound for the model checking problem of the logic of distributed knowing how. The exact complexity remains open. The main source of complexity lies in the recursive structure of A_G^*, while our lower bound already uses only Boolean combinations of simple formulas of the form $Kh_G p$. This suggests that the tight complexity may lie above DP. A natural next step is therefore to determine the exact complexity of the problem. Another interesting direction is to investigate the satisfiability problem for this logic.

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Logical Modeling of Belief Polarization

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This work-in-progress project is motivated by enhancing our theoretical understanding of the information exchange that leads to belief polarization. We approach this subject with the aim of using modal logic to model agents that update their beliefs based on the bias they already have. We propose a direction that builds on known formalisms in doxastic and epistemic logic where we add an update operation to an agents' strength of evidence, which leads to a stronger belief. This abstract briefly discusses these ideas and outlines objectives of on-going work.

1 Introduction

Polarization as a social phenomenon has been described and studied in many forms. Several sources point to different subcategorizations of the phenomenon, including, among others, *belief polarization*. According to [12], belief polarization occurs when people who disagree on a proposition individually strengthen their own beliefs when presented with a mixed body of evidence which both confirms and contradicts the belief they already hold. This should not be confused with group polarization, which also involves deliberation within the group, or with political polarization, which refers to division into ideological poles in democratic societies [12]. Recent works, such as [20, 18, 16], have developed modal logics to reason about group polarization, but little attention has yet been paid to exploring belief polarization at the individual level by utilizing the tools of epistemic and doxastic modal logics. In this project, we begin to bridge this gap by discussing ways of modeling information updates in which people update their beliefs with respect to the bias that they already have.

Our proposal uses several ingredients coming from dynamic epistemic logic [5], such as updating evidence in a setting based on evidence models [10], plausibility models [8, 9], and weighting models [4, 15]. In our setting, agents' beliefs are tied to the evidence they have at hand, thus changing the strength of evidence changes the agents' beliefs. We are interested in both quantitative and qualitative updates; in the following, we sketch a proposal for a quantitative update and discuss what a similar qualitative update could entail. The overall aim of this project is using modal logic to define fine-grained belief update mechanisms that can explain processes of belief polarization, where upon receiving new information, agents' beliefs are updated to a certain degree, based on their current evidence.

2 Updating Evidence

We begin by recalling the weighting models first introduced in [15] and further developed in [4]. Define At to be a countable set of atomic propositions. A **weighting model** is a structure $\mathscr{M} = (S, E, f, ||\bullet||)$

where S is a finite set of states; $E \subseteq \mathscr{P}(S)$ is a family of non-empty subsets $e \subseteq S$ ($\varnothing \notin E$) called evidence (sets) such that S is itself an evidence set ($S \in E$); $f : E \to \mathbb{N}$ is a function that assigns natural numbers to evidence sets; and $||\bullet|| : \mathrm{At} \to \mathscr{P}(S)$ is a valuation map. The intuition behind the weighting model is that it models what evidence (modeled as sets of possible worlds) an agent considers possible. The function f assigns a number to each evidence set denoting how convincing the evidence is to the agent. Consider the example in Figure 8. In this model, we have $S = \{w, v, s, t\}$ and evidence sets $E = \{e_1, e_2, e_3, e_4, S\}$ where the agent considers e_1 and e_2 the most convincing evidence and S the least convincing evidence according to the function f.

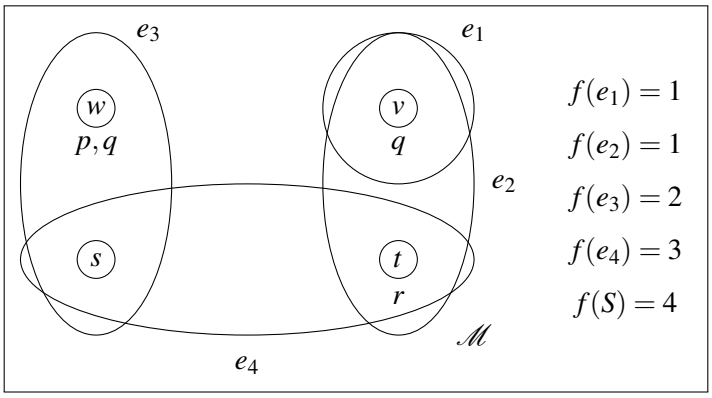

Figure 8: A weighting model $\mathscr{M}$

As defined in [4], the function f can be extended to the family of all bodies of evidence $\varepsilon \subseteq \mathscr{P}(E)$ such that $f(E) = \sum_{e \in E} f(e)$. Furthermore, define the **largest body of evidence** consistent with a given state $s \in S$, written $E_s := \{e \in E \mid s \in e\}$. Now, define the ordering for all $s, t \in S$: $s \leq_E t$ iff $f(E_s) \leq f(E_t)$. In [4], $\leq_E$ is shown to be a total plausibility ordering on S, meaning that the weighting model induces a total plausibility model as defined in [8, 9]. For all $s, t \in S$, let $s <_E t := s \leq_E t \,\&\, t \not\leq_E s$ and $s \simeq_E t := s \leq_E t \,\&\, t \leq_E s$. Returning to the example, note that $w \simeq_E v <_E t <_E s$, intuitively read as the agent considers states w and v equally most plausible, followed by t and s.

Following [9, 11], we define the **maximal elements** of a set $X \subseteq S$ denoted $Max_{\leq_E} X := \{s \in X \mid s \leq_E s' \text{ for all } s' \in X\}$. We also denote $P = ||p||$ as the truth set of $p \in \mathrm{At}$. Then, following [4] we can define the epistemic and doxastic notions: $KP := \{s \in S \mid P = S\}$; $BP := \{s \in S \mid Max_{\leq_E} S \subseteq P\}$; $B^Q P := \{s \in S \mid Max_{\leq_E} Q \subseteq P\}$; $S_b P := \{s \in S \mid P \neq \varnothing \text{ and } (\forall t \in P \text{ and } \forall w \notin P) : w <_E t\}$; and $K_D P := \{s \in S \mid t \not>_E s \Rightarrow t \in P\}$. We read KP as "the agent knows P", BP as "the agent believes P", $B^Q P$ as "the agent believes P conditional on Q", $S_b P$ as "the agent strongly believes P" and $K_D P$ as "the agent has defeasible knowledge that P".

Next, we propose a quantitative evidence update operation that is novel in this work. We are motivated to define a model update in which the agent updates their beliefs with respect to the bias that they already have. Thinking of weighted evidence as capturing the agent's biases, we can define an update on weighthing models that updates only the function f. Consider the updates $p\uparrow$ and $p\downarrow$ for a given atomic proposition p. The idea is that after an update $p\uparrow$, p becomes more convincing, whereas after an update $p\downarrow$, p becomes less convincing. In the light of belief polarization, we can think of $p\uparrow$ and $p\downarrow$ as positive and negative reactions to viewing some sort of documentation of p.

We define the updated model $\mathscr{M}^{p\uparrow} = (S, E, f^{p\uparrow}, ||\bullet||)$ where for all $e \in E$:[1]

[1] It is straightforward to generalize this definition where we exchange 1 with a given natural number $\theta \in \mathbb{N}$. This is a standard

$$f^{p\uparrow}(e) = \begin{cases} f(e)-1 & \text{if } \exists s \in e \text{ such that } s \in ||p|| \text{ and } f(e) > 1, \\ f(e) & \text{otherwise.} \end{cases}$$

Similarly, we define the updated model $\mathscr{M}^{p\downarrow} = (S, E, f^{p\downarrow}, ||\bullet||)$ where for all $e \in E$:

$$f^{p\downarrow}(e) = \begin{cases} f(e)+1 & \text{if } \exists s \in e \text{ such that } s \in ||p||, \\ f(e) & \text{otherwise.} \end{cases}$$

Returning to the example in Figure 8, we note that $P = \{w\}$, $Q = \{w, v\}$, and $R = \{t\}$. It follows that, for instance, for all $s \in S$, $s \in BQ$, but $s \notin BP$, since $Max_{\leq_E} S = \{w, v\} \subseteq Q$, but $Max_{\leq_E} S \not\subseteq P$. In other words, the agent believes Q, but not P. Now, consider the update $p\uparrow$. In the updated model $\mathscr{M}^{p\uparrow}$, f' is updated such that $f'(e_3) = f(e_3) - 1 = 1$, else $f'(e) = f(e)$. In turn, $f'(E_w) = 1$ while $f'(E_v) = f(E_v) = 2$, and thus this gives us a new updated plausibility order $w <'_E v <'_E t <'_E s$. It follows that $Max_{\leq'_E} S = \{w\}$ and hence in $\mathscr{M}^{p\uparrow}$, forall $s \in S : s \in BP$. In other words, after the update $p\uparrow$, the agent now believes p.

This update is quantitative in nature, as it leans on the definition of weighting models. To define a similar qualitative update, we exchange the function f in the weighting model with an ordering on the set E of evidence and then define an update that changes the ordering. A candidate for such a model is the justification model, also defined in [4], with appropriate refinements such that the update operations preserve the frame conditions. In on-going work, we provide a dynamic language with epistemic and doxastic operators, as well as updates $[p\uparrow]\varphi$ and $[p\downarrow]\varphi$, in order to obtain a sound and complete axiomatic system with respect to the class of weighting models.

This project situates itself at the intersection of two bodies of related literature: work on formal models of belief polarization and epistemic and doxastic logics with belief updates within groups, based on evidence. Much of the existing formal models of belief polarization in cognitive science uses a probabilistic Bayesian approach. An example is [17] which utilizes Bayesian networks to model when it is rational for reasoners with opposing beliefs to both strengthen their beliefs, both weaken their beliefs or for one to strengthen and the other to weaken their belief. It is shown that in some instances, belief polarization is consistent with rational belief updates. Such findings can tie in with results obtained in logic-based models of informational cascades [1, 3], known to be phenomena where people are rational to ignore their own information and follow the observed behavior by others. In [3], a Bayesian framework is cast into a probabilistic logical model where both the agents' beliefs and knowledge is modeled. It is shown that even when agents are rational reasoners and have unlimited power of higher-order knowledge of other agents' knowledge, these types of cascades can not always be prevented. Another Bayesian formalism of belief polarization is put forward in [14] and models cases where agents can rationally predict that they will participate in belief polarization. In a logical framework this type of reasoning would ideally require a higher-order notion of belief, namely the prediction of belief. Incorporating these ideas in our presentation will require an extended setting along the lines of the probabilistic logical approach in [1, 3] to model higher-order beliefs about participating in belief polarization. Our project has adopted a specific approach to represent notions of evidence in relation to belief and knowledge, other choices that could have been made include e.g. the combination of DEL with justification logic in [6, 7], with a 4-valued logic approach in [19] or with topological models in [2].

approach in threshold models for social influence, seen in a modal logic context in work such as [13]. For now, we keep 1 as it represents the smallest possible change in f considering that its range is the set of natural numbers.

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