On cut-elimination for the Grzegorczyk modal logic

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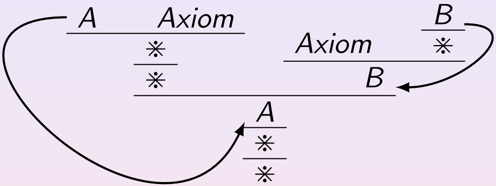
Cyclic and non-well-founded proofs



Proof systems allowing non-well-founded reasoning can be defined for

- modal μ-calculus
- Lambek calculus with iteration
- Peano arithmetic
- GL, Grz, K^+, etc.

Grzegorczyk modal logic Grz

Axiom schemas:

- Boolean tautologies;
- $\Box(A \to B) \to (\Box A \to \Box B)$;
- $\Box(\Box(A \to \Box A) \to A) \to A$.

Inference rules:

$$\text{mp}\ \frac{A \qquad A \to B}{B}\ , \qquad \text{nec}\ \frac{A}{\Box A}\ .$$

The Grzegorczyk modal logic Grz can be characterized by reflexive partially ordered Kripke frames without infinite ascending chains.

Non-well-founded sequent calculus Grz_∞

Axioms and inference rules

$$\Gamma, p \Rightarrow p, \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

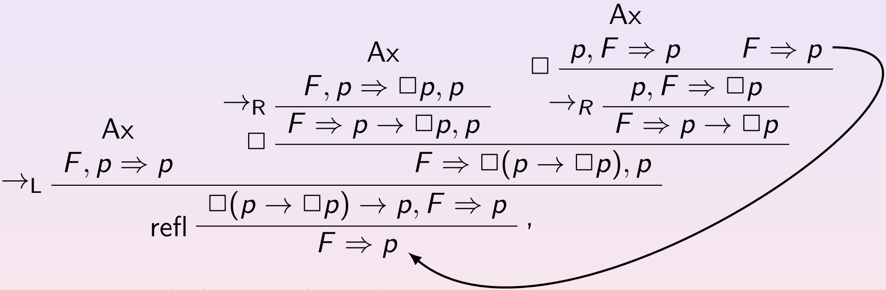
$$\to_L \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_R \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

$$\mathsf{refl} \frac{\Gamma, B, \Box B \Rightarrow \Delta}{\Gamma, \Box B \Rightarrow \Delta}, \qquad \Box \frac{\Gamma, \Box\Pi \Rightarrow A, \Delta \qquad \Box\Pi \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}.$$

Global condition: every infinite branch in an ∞–proof must pass through a right premise of the rule ($\Box$) infinitely many times.

Example

A non-well-founded proof



where $F = \Box(\Box(p \to \Box p) \to p)$.

Theorem. For any finite multisets of formulas Γ and Δ, we have

$$\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta \iff \mathsf{Grz} \vdash \bigwedge \Gamma \to \bigvee \Delta.$$

Non-well-founded sequent calculus $\mathsf{Grz}_\infty + \mathsf{cut}$

System with cut

The system $\mathsf{Grz}_\infty + \mathsf{cut}$ is obtained from Grz_∞ by adding the cut rule

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ .$$

Global condition remains to be the same.

Cut elimination procedure?

Obviously, we can try to push applications of the cut rule away from the root. But is this procedure productive? If so, why would the resulting sequent tree satisfy the global condition for infinite branches?

The required cut-elimination operator can be defined by coinduction with subinduction on certain parameters.

We want to treat nested inductive-co-inductive definitions in a general way.

So we adopt an approach from denotational semantics of computer languages, where recursive equations obtained from inductive-co-inductive definitions are solved via fixed-point theorems.

Cut elimination for finite proofs

A possible strategy for finite proofs

- Given two cut-free proofs for premises of the cut rule, define a cut-free proof of the conclusion.
- Given a proof with cut, apply the procedure of the previous point top-down (by induction on the height of the given proof).

The case of non-well-founded proofs

For non-well-founded proofs we can

- Write out recursive equations for an operator eliminating one root application of the cut rule.
- Using a solution of the previous step, write out recursive equations for a full cut elimination operator.

The *n-fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every branch at the nth from the root right premise of the rule $(\Box)$, so that this premise is removed.

The 1-fragment of an ∞–proof is also called its *main fragment*.

$$
\text{refl}\;\dfrac{\to_L\;\dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow p} \qquad \Box\;\dfrac{\to_R\;\dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p} \qquad \to_R\;\dfrac{\Box\;\dfrac{\dfrac{\text{Ax}}{p, F \Rightarrow p} \qquad \dfrac{\vdots}{F \Rightarrow p}}{p, F \Rightarrow \Box p}}{F \Rightarrow p \to \Box p}}{F \Rightarrow \Box(p \to \Box p), p}}{\Box(p \to \Box p) \to p, F \Rightarrow p}}{F \Rightarrow p}\,,
$$

where $F = \Box(\Box(p \to \Box p) \to p)$.

We define the *local height $|\pi|$ of an ∞–proof π* as the length of the longest branch in its main fragment.

The *n-fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every branch at the nth from the root right premise of the rule ($\Box$), so that this premise is removed.

The 1-fragment of an ∞–proof is also called its *main fragment*.

$$
\text{refl}\ \dfrac{\to_L\ \dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow p} \qquad \Box\ \dfrac{\to_R\ \dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p} \qquad \to_R\ \dfrac{\Box\ \dfrac{\dfrac{\text{Ax}}{p, F \Rightarrow p}}{p, F \Rightarrow \Box p}}{F \Rightarrow p \to \Box p}}{F \Rightarrow \Box(p \to \Box p), p}}{\Box(p \to \Box p) \to p, F \Rightarrow p}}{F \Rightarrow p},
$$

where $F = \Box(\Box(p \to \Box p) \to p)$.

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The *n-fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every branch at the nth from the root right premise of the rule $(\Box)$, so that this premise is removed.

The 1-fragment of an ∞–proof is also called its *main fragment*.

$$\text{refl}\;\dfrac{\to_{\mathsf{L}}\;\dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow p} \qquad \Box\;\dfrac{\to_{\mathsf{R}}\;\dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p}}{F \Rightarrow \Box(p \to \Box p), p}}{\Box(p \to \Box p) \to p, F \Rightarrow p}}{F \Rightarrow p}\;,$$

where $F = \Box(\Box(p \to \Box p) \to p)$.

We define the *local height $|\pi|$ of an ∞–proof π* as the length of the longest branch in its main fragment.

We write $\pi \sim_n \tau$ if n-fragments of ∞-proofs π and τ coincide. For any π and τ, we also set $\pi \sim_0 \tau$.

The set of ∞-proofs $\mathcal{P}$ is a (spherically) complete (ultra)metric space, where the distance function is defined by

$$d_{\mathcal{P}}(\pi, \tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

We see that $d_{\mathcal{P}}(\pi, \tau) \leqslant 2^{-n}$ if and only if $\pi \sim_n \tau$. Thus, the ultrametric $d_{\mathcal{P}}$ can be considered as a measure of similarity between ∞-proofs.

Ultrametric spaces

A metric space (M, d) is ultrametric if it satisfies a stronger version of the triangle inequality:

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\}.$$

An ultrametric space is spherically complete if an arbitrary descending sequence of closed balls has a common point.

A function $f : M \to M$ is contractive if $d(f(x), f(y)) < d(x, y)$ when $x \neq y$.

Theorem (Prieß-Crampe 1990, Petalas and Vidalis 1993)

Any contractive mapping on a non-empty spherically complete ultrametric space has a unique fixed-point.

In an ultrametric space (M, d), a function $f\colon M \to M$ is called *non-expansive* if $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in M$.

For ultrametric spaces (M, d_M) and (N, d_N), the Cartesian product $M \times N$ can be also considered as an ultrametric space with the metric

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) = \max\{d_M(x_1, x_2), d_N(y_1, y_2)\}.$$

Hence, for any $m \in \mathbb{N}$, we have an ultrametric on the set M^m.

Note that any function $\mathrm{u}\colon \mathcal{P}^m \to \mathcal{P}$ is non-expansive if and only if for any tuples $\vec{\pi}$ and $\vec{\pi}'$, and any $n \in \mathbb{N}$ we have

$$\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m \Rightarrow \mathrm{u}(\vec{\pi}) \sim_n \mathrm{u}(\vec{\pi}').$$

For $m \in \mathbb{N}$, let $\mathcal{F}_m$ denote the set of all non-expansive functions from $\mathcal{P}^m$ to $\mathcal{P}$.

We introduce an ultrametric for $\mathcal{F}_m$ in the following way. For $\mathsf{a}, \mathsf{b} \in \mathcal{F}_m$, we write

$$\mathsf{a} \sim_{n,k} \mathsf{b} \iff \begin{cases} \mathsf{a}(\vec{\pi}) \sim_n \mathsf{b}(\vec{\pi}) & \text{for any } \vec{\pi} \in \mathcal{P}^m, \\ \mathsf{a}(\vec{\pi}) \sim_{n+1} \mathsf{b}(\vec{\pi}) & \text{whenever } \pi \text{ with } \sum_{i=1}^{m} |\pi_i| < k. \end{cases}$$

The ultrametric l_m is defined by

$$l_m(\mathsf{a}, \mathsf{b}) = \frac{1}{2} \inf \left\{ \frac{1}{2^n} + \frac{1}{2^{n+k}} \mid \mathsf{a} \sim_{n,k} \mathsf{b} \right\}.$$

Notice that any operator $U\colon \mathcal{F}_m \to \mathcal{F}_m$ is strictly contractive if and only if for any $a, b \in \mathcal{F}_m$, and any $n, k \in \mathbb{N}$ we have

$$a \sim_{n,k} b \Rightarrow U(a) \sim_{n,k+1} U(b).$$

Proposition

Every strictly contractive mapping $U\colon \mathcal{F}_m \to \mathcal{F}_m$ has a unique fixed-point.

Proposition

For any formula A, there exists a non-expansive mapping $\mathrm{re}_A\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ such that

- $\mathrm{re}_A(\pi, \tau)$ is an ∞-proof of the conclusion of the cut-rule with the cut formula A whenever π and τ are ∞-proofs for premises;
- the n-fragment of $\mathrm{re}_A(\pi, \tau)$ doesn't contain applications of the cut rule whenever n-fragments of π and τ do so.

A mapping $\mathrm{u}\colon \mathcal{P} \to \mathcal{P}$ is called *root-preserving* if it maps ∞-proofs to ∞-proofs of the same sequents. Let $\mathcal{T}$ denote the set of all root-preserving non-expansive mappings from $\mathcal{P}$ to $\mathcal{P}$.

Proposition

In the ultrametric space $(\mathcal{T}, l_1)$, any contractive operator $\mathrm{U} : \mathcal{T} \to \mathcal{T}$ has a unique fixed-point.

We construct the required cut-elimination mapping ce so it commutes with every application of inference rules except (cut) and satisfies the following condition:

$$\mathrm{ce}\left(\text{cut}\,\frac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Gamma, A \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}\right) = \mathrm{re}_A(\mathrm{ce}(\pi_0), \mathrm{ce}(\pi_1)).$$

In order to do this, we define a contractive operator $\mathrm{F}\colon \mathcal{T} \to \mathcal{T}$ and obtain the mapping ce as the fixed-point of F.

For a mapping $u \in \mathcal{T}$ and an ∞-proof π, the ∞-proof $F(u)(\pi)$ is defined as follows. If $|\pi| = 0$, then we define $F(u)(\pi)$ to be π. Otherwise, we define $F(u)(\pi)$ according to the last application of an inference rule in π:

$$\rightarrow_L \frac{\overset{\pi_0}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\pi_1}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \rightarrow B \Rightarrow \Delta} \longmapsto \rightarrow_L \frac{\overset{u(\pi_0)}{\Gamma, B \Rightarrow \Delta} \qquad \overset{u(\pi_1)}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \rightarrow B \Rightarrow \Delta},$$

$$\rightarrow_R \frac{\overset{\pi_0}{\Gamma, A \Rightarrow B, \Delta}}{\Gamma \Rightarrow A \rightarrow B, \Delta} \longmapsto \rightarrow_R \frac{\overset{u(\pi_0)}{\Gamma, A \Rightarrow B, \Delta}}{\Gamma \Rightarrow A \rightarrow B, \Delta},$$

$$\mathrm{refl} \frac{\overset{\pi_0}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta} \longmapsto \mathrm{refl} \frac{\overset{u(\pi_0)}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta},$$

$$\Box\,\dfrac{\begin{matrix}\pi_0 \\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \longmapsto \Box\,\dfrac{\begin{matrix}\mathsf{u}(\pi_0) \\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\mathsf{u}(\pi_1) \\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\,,$$

$$\mathsf{cut}\,\dfrac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Gamma, A \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta} \longmapsto \mathsf{re}_A(\mathsf{u}(\pi_0), \mathsf{u}(\pi_1)).$$

Now the operator F is well-defined. By the case analysis according to the definition of F, we see that F(u) is non-expansive and belongs to $\mathcal{T}$ whenever $\mathsf{u} \in \mathcal{T}$. In addition, $\mathsf{F} \colon \mathcal{T} \to \mathcal{T}$ is contractive.

Theorem

If $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$.

The modal logic of transitive closure

Modal logic of transitive closure K^+

Axiom schemas:

- Boolean tautologies;
- $\Box(A \to B) \to (\Box A \to \Box B)$;
- $\Box^+(A \to B) \to (\Box^+ A \to \Box^+ B)$;
- $\Box^+ A \to \Box A \wedge \Box\Box^+ A$;
- $\Box A \wedge \Box^+(A \to \Box A) \to \Box^+ A$.

Inferences rules:

$$\text{mp}\ \frac{A \qquad A \to B}{B}\,, \qquad \text{nec}\ \frac{A}{\Box^+ A}\,.$$

The modal logic K^+ can be characterized by the class of Kripke frames of the form (W, R, R^+).

Non-well-founded sequent calculus K^+_∞

Axioms and inference rules

$$\Gamma, p \Rightarrow p, \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

$$\to_L \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_R \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

Non-well-founded sequent calculus K^+_∞

$$\Box \ \frac{\Sigma, \Pi, \Box^+\Pi \Rightarrow A}{\Gamma, \Box\Sigma, \Box^+\Pi \Rightarrow \Box A, \Delta}\ ,$$

$$\Box^+ \ \frac{\Sigma, \Pi, \Box^+\Pi \Rightarrow A \qquad \Sigma, \Pi, \Box^+\Pi \Rightarrow \Box^+ A}{\Gamma, \Box\Sigma, \Box^+\Pi \Rightarrow \Box^+ A, \Delta}\ .$$

Global condition: any infinite branch contains a thread of a formula $\Box^+ A$ that passes through the right part of the right premise of the rule ($\Box^+$) infinitely many times.

Equivalently, every infinite branch in an ∞-proof must contain a tail with the following properties: all applications of the rule ($\Box^+$) in the tail have the same principal formula $\Box^+ A$; the tail passes through the right premise of the rule ($\Box^+$) infinitely many times; the tail doesn't pass through a left premise of the rule ($\Box^+$); there are no applications of the rule ($\Box$) in the tail.

Example

A non-well-founded proof

$$\Box^+\;\dfrac{\overset{\text{Ax}}{p, F, \Box^+F \Rightarrow p} \qquad \to_{\mathsf{L}}\;\dfrac{p, \Box p, \Box^+F \Rightarrow \Box^+p \qquad \overset{\text{Ax}}{p, \Box^+F \Rightarrow p, \Box^+p}}{p, F, \Box^+F \Rightarrow \Box^+p}}{p, \Box p, \Box^+F \Rightarrow \Box^+p}\;.$$

where $F = p \to \Box p$.

Example

A non-well-founded proof

$$
\Box^+ \dfrac{\dfrac{\text{Ax}}{p, F, \Box^+ F \Rightarrow p} \qquad \to_L \dfrac{p, \Box p, \Box^+ F \Rightarrow \color{blue}{\Box^+ p} \qquad \dfrac{\text{Ax}}{p, \Box^+ F \Rightarrow p, \Box^+ p}}{p, F, \Box^+ F \Rightarrow \color{blue}{\Box^+ p}}}{p, \Box p, \Box^+ F \Rightarrow \color{blue}{\Box^+ p}} .
$$

where $F = p \to \Box p$.

Example

A non-well-founded proof

$$
\dfrac{\dfrac{\text{Ax}}{p, F, \Box^+ F \Rightarrow p} \qquad \dfrac{\dfrac{}{p, \Box p, \Box^+ F \Rightarrow \Box^+ p} \qquad \dfrac{\text{Ax}}{p, \Box^+ F \Rightarrow p, \Box^+ p}}{p, F, \Box^+ F \Rightarrow \Box^+ p}\,{\to_{\mathsf{L}}}}{p, \Box p, \Box^+ F \Rightarrow \Box^+ p}\,\Box^+ \; .
$$

where $F = p \to \Box p$.

Theorem (see Niwiński and Walukiewicz 1996, Bucheli, Kuznets and Studer 2010, Doczkal and Smolka 2012)

The system K^+_∞ is a sound and complete deductive system for the modal logic of transitive closure K^+.

Non-well-founded sequent calculus $\mathsf{K}^+_\infty + \mathsf{cut}$

System with cut

The system $\mathsf{K}^+_\infty + \mathsf{cut}$ is obtained from K^+_∞ by adding the cut rule

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ .$$

Global condition remains to be the same.

Non-well-founded sequent calculus $\mathsf{K}^+_\infty + \mathsf{cut}$

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Global condition remains to be the same.

Obviously, we can push applications of the cut rule away from the root. It can be shown that this procedure will be productive and will define a cut-free sequent tree.

Non-well-founded sequent calculus $\mathsf{K}^+_\infty + \mathsf{cut}$

System with cut

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$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ .$$

Global condition remains to be the same.

Obviously, we can push applications of the cut rule away from the root. It can be shown that this procedure will be productive and will define a cut-free sequent tree.

But why will this tree satisfy the global condition on infinite branches?

The required cut-elimination operator can be defined by induction with subcoinduction with subsubinduction on certain parameters.

These parameters are

- global height $\|\pi\|$ of a non-well-founded proof π
- *n*-similarity between non-well-founded proofs
- local height $|\pi|$ of a non-well-founded proof π

The set of non-well-founded proofs is a (spherically) complete (ultra)metric space, where the distance function is defined by

$$d(\pi, \tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

Analogously to the case of Grz, on the set of operators on non-well-founded proofs, a spherically complete generilized ultrametric structure can be defined via the following similarity relations

$$u \sim_{\alpha,n,k} v \iff \begin{cases} u(\pi) = v(\pi) & \text{for any } \pi \text{ with } \|\pi\| < \alpha, \\ u(\pi) \sim_n v(\pi) & \text{for any } \pi \text{ with } \|\pi\| = \alpha, \\ u(\pi) \sim_{n+1} v(\pi) & \text{for any } \pi \text{ with } \|\pi\| = \alpha \text{ and } |\pi| < k. \end{cases}$$

Recursive equations for a full cut elimination operator

Given a non-expansive operator re_A eliminating any root application of the cut rule with the cut formula formula A, we write out recursive equations for a full cut-elimination operator ce.

An operator ce maps axioms to the same axioms and commutes with applications of all inference rules except the cut rule.

$$\mathsf{ce}\left(\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma, A \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}\right) = \mathsf{re}_A(\mathsf{ce}(\pi_1), \mathsf{ce}(\pi_2))$$

Thank you!