Frugal Preference Aggregation: The Ex-Ante Condorcet Approach

Klaus Nehring and Clemens Puppe

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Agenda

1. Introduction
2. The Ex-Ante Condorcet Approach
3. Plain Convexity
4. Symmetric Priors: The Tukey Median
Motivation

- Majoritarian choice under ‘ambiguous’ beliefs about voters’ prefs.
- **Frugal aggregation model:**
  - Information about individual top choices.
  - Ambiguous beliefs about underlying complete preferences.
- **Ex-ante Condorcet approach:**
  - Every pair of alternatives induces possible expected majorities...
  - ... and thus yields ex-ante net majority tournament.
  - Complete ignorance about ex-post preferences.
  - Maximal elements: ‘Ex-ante Condorcet winners.’
- Applications: budget allocation (‘participatory budgeting’), spatial voting, collective choice in space of characteristics.
- Different assumptions on the epistemic state of social evaluator:
  - Plain convexity
  - Symmetry
Motivation

- Majoritarian choice under ‘ambiguous’ beliefs about voters’ prefs.

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  - Maximal elements: ‘Ex-ante Condorcet winners.’

- Applications: budget allocation (‘participatory budgeting’), spatial voting, collective choice in space of characteristics.

- Different assumptions on the epistemic state of social evaluator:
  - Plain convexity $\rightarrow$ generic plurality winner.
  - Symmetry $\rightarrow$ (strict) Tukey median.
Relation to Literature

- Frugal aggregation as opposed to
  (i) standard *Arrovian preference aggregation* on economic domains (Le Breton & Weymark, 2004), and
  (ii) standard *spatial voting* (Austen-Smith & Banks, 1999) (which is a degenerate special case).

- (Non-Bayesian) preference aggregation under incomplete information (Boutilier & Rosenschein, 2016; Lang, 2020).

- Decision making under complete ignorance (Luce & Raiffa, 1957; Nehring, 2000; 2009).

- Participatory budgeting (Aziz & Shah, 2020, Goel et al., 2019; Freeman et al., 2021).
Background: Pref. Aggregation on Economic Domains

- Consider a *common* restricted domain $D_i = D \subseteq \mathcal{R}$ on some set of alternatives $X$, where $\mathcal{R}$ is the set of all weak orders.

- $D$ has the **free triple** property if the restriction of $D$ to any triple of distinct alternatives is unrestricted.

- **Proposition** If $D$ has the free triple property, then every social welfare function $F : D^n \rightarrow \mathcal{R}$ satisfying IIA and WP is dictatorial.
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**Proposition** If \( D \) has the free triple property, then every social welfare function \( F : D^n \rightarrow \mathcal{R} \) satisfying IIA and WP is dictatorial.

**Example:** Let \( X \subseteq \mathbb{R}^L \) be convex, and consider the space of all convex preferences \( D_{co} \) on \( X \). This domain does **not** have the free triple property.
Consider a *common* restricted domain $\mathcal{D}_i = \mathcal{D} \subseteq \mathcal{R}$ on some set of alternatives $X$, where $\mathcal{R}$ is the set of all weak orders.

$\mathcal{D}$ has the **free triple** property if the restriction of $\mathcal{D}$ to any triple of distinct alternatives is unrestricted.

**Proposition** If $\mathcal{D}$ has the free triple property, then every social welfare function $F : \mathcal{D}^n \rightarrow \mathcal{R}$ satisfying IIA and WP is dictatorial.

**Example:** Let $X \subseteq \mathbb{R}^L$ be convex, and consider the space of all convex preferences $\mathcal{D}_{co}$ on $X$. This domain does **not** have the free triple property.

**Proposition** Suppose $X \subseteq \mathbb{R}^L$, and for all distinct $a, b \in X$, there is $c \in X$ such that $\{a, b, c\}$ is not collinear (note: $\Rightarrow L > 1$); then, every social welfare function $F : \mathcal{D}_{co}^n \rightarrow \mathcal{R}$ satisfying IIA and WP is dictatorial (follows from Kalai, Muller & Satterthwaite, 1979).
Background: (No) Condorcet Winners in Spatial Voting

- Standard spatial voting assumes that $X \subseteq \mathbb{R}^L$ and that voters have **Euclidean preferences**, i.e. a voter with top $\theta_i$ has utility function

  $$u_{\theta_i}(x) = -(||x - \theta_i||_2)^2 = -(x - \theta_i)^T \cdot (x - \theta_i).$$

  Geometrically, voters have *circles* as indifference curves.
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- Condorcet winners?
  \[
  \theta_2 \triangleright \text{Maj} \theta_3 \triangleright \text{Maj} \theta_1
  \]

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- Condorcet winners?

\[
\begin{align*}
\theta_2 \\
& \quad b \\
& \quad \quad \bullet \\
& \quad \quad \bullet \quad c \\
& \quad \bullet \quad \bullet \quad \bullet \theta_3 \\
\theta_1
\end{align*}
\]
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\[ b \bullet \]
\[ c \bullet \]
\[ a \bullet \]
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\[
\begin{array}{c}
\theta_2 \bullet \\
\theta_1 \\
\theta_3 \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
b \\
c \\
a \\
\end{array}
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- Condorcet winners?

Thus, $a \succ_{\text{Maj}} b$, $b \succ_{\text{Maj}} c$, $c \succ_{\text{Maj}} a$!
Background: (No) Condorcet Winners in Spatial Voting

- Hence, in the above situation, every element of \( \{a, b, c\} \) beats any other element of \( \{a, b, c\} \) via a majority path.

- Proposition (McKelvey, 1976) Generically, every \( x \in X \) beats every \( y \in X \) along some majority path in the standard spatial voting model.

- This negative result has been significantly generalized to the case of general continuous preferences and voting procedures beyond pairwise majority comparisons by McKelvey (1979).

- Upshot: In spatial voting, Condorcet winners fail to exist almost always. And even worse, generically an agenda setter can induce every alternative as the winner of a sequential majority vote by choosing an appropriate sequence of intermediate comparisons (‘McKelvey-Schofield Chaos Theorem’).
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The Ex-Ante Condorcet Approach

- $X = \{x, y, \ldots\}$ abstract space of alternatives.
- $\theta = (\theta_1, \ldots, \theta_n) \in X^n$ known profile of (unique) top choices.
- $\pi$ prior belief over underlying profile $(\succeq_1, \ldots, \succeq_n)$ of complete (‘ex-post’) preferences on $X$.
- $\Pi$ convex set of priors: social evaluator’s ambiguous beliefs.
  - E.g. $X \subseteq \mathbb{R}^L$, $\Pi_{co}$ all priors over profiles of convex preferences.
- $(\theta, \Pi)$ epistemic state of social evaluator.
- Every $\pi$ induces expected majority count for all $x, y \in X$:
  $$m_{(\theta, \pi)}(x, y) := E_\pi[\#\{i : x \succeq_i y\}].$$
- Every ambiguous belief $\Pi$ thus induces interval of possible expected majority margins $\left[m^-_{(\theta, \Pi)}(x, y), m^+_{(\theta, \Pi)}(x, y)\right]$. 

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**Ex-Ante Condorcet Winners**

- The **ex-ante majority relation** is defined by

\[ xR_{(\theta, \Pi)} y \iff m^-_{(\theta, \Pi)}(x, y) \geq m^-_{(\theta, \Pi)}(y, x) \]

\[ \iff m^+_{(\theta, \Pi)}(x, y) \geq m^+_{(\theta, \Pi)}(y, x). \]

- Majority intervals may overlap but are unambiguously ordered.
  - Choice based on \( \alpha \min + (1 - \alpha) \max \) (‘Hurwicz pessimism – optimism index,’ Luce & Raiffa) independent of \( \alpha \).
  - Independent of ambiguity attitude.

- **Ex-ante Condorcet winners:**

\[ CW(\theta, \Pi) := \{ x \in X | xR_{(\theta, \Pi)} y \text{ for all } y \in X \}. \]

- **Main finding:** in interesting cases, ex-ante CWs exist even if ex-post CWs fail to exist (e.g. in standard spatial voting).
Simple Examples

- **Unrestricted beliefs:** $\Pi_{\text{univ}}$
  - $x P_{(\theta, \Pi)} y$ iff more agents have top at $x$ than top at $y$.
  - $CW(\theta, \Pi) = \text{plurality winners.}$
Simple Examples

- **Unrestricted beliefs**: $\Pi_{\text{univ}}$
  - $xP_{(\theta, \Pi)}y$ iff more agents have top at $x$ than top at $y$.
  - $\text{CW}(\theta, \Pi) = \text{plurality winners.}$

- **Convexity** (‘single-peakedness’) on the line: $\Pi_{\text{co}}$ on $X = \mathbb{R}$
  - For $x < y$,
    - $xP_{(\theta, \Pi)}y$ iff more agents have top in $(-\infty, x]$ than in $[y, +\infty)$.
  - $\text{CW}(\theta, \Pi) = \text{medians.}$
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The Multi-Dimensional Problem

Now, $X$ convex subset of $\mathbb{R}^L$, e.g.

$$X_{\text{res}} = \left\{ x = (x^1, \ldots, x^L) \in \mathbb{R}^L : \sum_{\ell=1}^{L} x^\ell = M \right\}$$
Now, $X$ is a convex subset of $\mathbb{R}^L$, e.g.

$$X_{\text{res}} = \left\{ x = (x^1, \ldots, x^L) \in \mathbb{R}^L : \sum_{\ell=1}^{L} x^\ell = M \right\}$$
Convexity of Preferences

- Suppose \( \Pi_{\text{co}} \) consists of all possible priors on profiles of individually convex preferences (‘plain convex model’).

**Proposition**

*If individual tops \( \{\theta_1, \ldots, \theta_n\} \) are in general position (no three collinear), then \( \text{CW}(\theta, \Pi_{\text{co}}) \) contains all plurality winners. Moreover, if there is a unique plurality winner \( \theta_i^* \), then \( \text{CW}(\theta, \Pi_{\text{co}}) = \{\theta_i^*\} \).*
Introduction  The Ex-Ante Condorcet Approach  Plain Convexity  Symmetric Priors: The Tukey Median

**Conundrum**

$U$

$y$

$x = \theta_1 = \theta_2$

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Conundrum

- Tops randomly drawn from $U$ (in particular, in general position).
- Convexity gives no info about pref between $y$ and $x$.
- With duplication, ex-ante Condorcet winner is $x$.
- In fact, possibly all voters with top in $U$ prefer $x$ to $y$.
- But alternative $x$’s claim for winner is like ‘grasping at straws.’
- Indeed, prefs must be very specifically and individually tailored.
Introduction
The Ex-Ante Condorcet Approach
Plain Convexity
Symmetric Priors: The Tukey Median

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Separating Top from Preference Structure

- **Intuition:** tops contain no info about preference ‘structure.’
- **Flexible workhorse:** quadratic preferences, i.e.,

  \[ u_{\theta_i}(x) = -(x - \theta_i)^T \cdot Q \cdot (x - \theta_i) \]

  with positive definite \( Q \).

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\[ \begin{align*}
\bullet & \quad \text{tops preferring } x \text{ to } y \\
\bullet & \quad \text{tops preferring } y \text{ to } x 
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with positive definite \( Q \).

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Frugal Preference Aggregation: The Ex-Ante Condorcet Approach
Proposition

\( \Pi_{\text{quad}} \) set of all priors on profiles of quadratic preferences.

**Proposition**

\( \Pi_{\text{quad}} \) induces same ex-ante majority relation as \( \Pi_{\text{co}} \). In particular,

\[ CW(\theta, \Pi_{\text{quad}}) = CW(\theta, \Pi_{\text{co}}). \]
The Symmetric Quadratic Model

- Quadratic model allows one to formulate above intuition that “tops contain no info about preference structure” as follows:
- A prior $\pi$ on profiles of quadratic preferences is **symmetric** if, for all $i, j$,
  \[ \pi Q_i = \pi Q_j. \]
- $\Pi_{\text{sym quad}}$ set of all **symmetric** priors on profiles of quadratic preferences.

**Question**

$CW(\theta, \Pi_{\text{quad}}^{\text{sym}})$?
The Tukey Median

- *Tukey depth* of $x$ at $\theta$: $\delta(x, \theta) := \min_{H \ni x} \# (\theta \cap H)$.
- Let $\delta(\theta) := \max_{x \in X} \delta(x, \theta)$.
- The **Tukey median** (Tukey, 1975) is defined by

  $$T(\theta) := \{x \in X : \delta(x, \theta) = \delta(\theta)\}.$$
The Tukey Median

- **Tukey depth** of $x$ at $\theta$: $\varrho(x, \theta) := \min_{H \ni x} \#(\theta \cap H)$.
- Let $\varrho(\theta) := \max_{x \in X} \varrho(x, \theta)$.
- The **Tukey median** (Tukey, 1975) is defined by

$$T(\theta) := \{x \in X : \varrho(x, \theta) = \varrho(\theta)\}.$$
The Tukey Median

- **Tukey depth** of \( x \) at \( \theta \): \( \vartheta(x, \theta) := \min_{H \ni x} \#(\theta \cap H) \).
- Let \( \vartheta(\theta) := \max_{x \in X} \vartheta(x, \theta) \).
- The **Tukey median** (Tukey, 1975) is defined by
  \[
  T(\theta) := \{ x \in X : \vartheta(x, \theta) = \vartheta(\theta) \}.
  \]

- Tukey median is one (affinely invariant) multi-dimensional median see, e.g., survey by Rousseeuw & Hubert (2017).
Example: The Pentagon

\[ \theta_1 \]

\[ \theta_2 \]

\[ \theta_3 \]

\[ \theta_4 \]

\[ \theta_5 \]
Example: The Pentagon
Example: The Pentagon
Denote by $\mathcal{H}_x^* := \{ H \ni x : \#(\theta \cap H) = \theta(\theta) \}$ and let

$$T^*(\theta) := \{ x \in T(\theta) \mid \text{for no } y \in T(\theta), \mathcal{H}_y^* \subsetneq \mathcal{H}_x^* \}.$$ 

By construction, $T^*(\theta) \subseteq T(\theta)$. 

$T^*(\theta)$ strict Tukey median.
Strict Tukey Median and Main Result

- Denote by $\mathcal{H}_x^* := \{H \ni x : \#(\theta \cap H) = \vartheta(\theta)\}$ and let
  
  $$T^*(\theta) := \{x \in T(\theta) \mid \text{for no } y \in T(\theta), \mathcal{H}_y^* \subsetneq \mathcal{H}_x^*\}.$$  

- By construction, $T^*(\theta) \subseteq T(\theta)$.
- $T^*(\theta)$ strict Tukey median.

**Theorem**

*For all profiles $\theta$, the strict Tukey median is non-empty and

$$\text{CW}(\theta, \Pi_{\text{sym}}^{\text{quad}}) = T^*(\theta).$$*
**Sketch of Proof**

**Step 1**

The symmetric quadratic model $\Pi_{\text{quad}}^{\text{sym}}$ induces the same ex-ante majority relation as the **uniform** quadratic model $\Pi_{\text{quad}}^{\text{unif}}$ consisting of all priors that assign full mass to uniform profiles

$$[(\theta_1, Q), (\theta_2, Q), \ldots, (\theta_n, Q)]$$

for some **common** quadratic from $Q$. 
Sketch of Proof

Step 2

This implies that, under symmetry, the ex-ante majority relation coincides locally with relative Tukey depth:

\[ xR_{\theta,Tuk}y \iff \min_{H \ni x, H \not\ni y} #(\theta \cap H) \geq \min_{H \ni y, H \not\ni x} #(\theta \cap H). \]
Sketch of Proof

**Step 2**

This implies that, under symmetry, the ex-ante majority relation coincides locally with **relative Tukey depth**:

\[ x \overset{R_{(\theta,Tuk)}}{\sim} y \iff \min_{H \ni x, H \not\ni y} \#(\theta \cap H) \geq \min_{H \ni y, H \not\ni x} \#(\theta \cap H). \]

Observe that this solves above conundrum:

\[ x = \theta_1 = \theta_2 \]

\[ y \]
Step 2

This implies that, under symmetry, the ex-ante majority relation coincides locally with relative Tukey depth:

\[ x R_{(\theta, \text{Tuk})} y \iff \min_{H \ni x, H \not\ni y} \#(\theta \cap H) \geq \min_{H \ni y, H \not\ni x} \#(\theta \cap H). \]

Observe that this solves above conundrum:
### Sketch of Proof

#### Step 3

*Now show that the strict Tukey median consists of the maximal elements of the relative Tukey depth:*

\[ T^*(\theta) = \{ x \in X : xR(\theta, \text{Tuk}) y \text{ for all } y \in X \}. \]

#### Step 4

*Finally, show that \( \{ x \in X : xR(\theta, \text{Tuk}) y \text{ for all } y \in X \} \) is non-empty by appeal to Zorn’s Lemma.*
Example: Relative vs. Absolute Tukey Depth

\[ \theta_1 \]
\[ \theta_5 \]
\[ \theta_4 \]
\[ \theta_3 \]
\[ \theta_2 \]

Minimal depth of \( x \) relative to \( y \): \( \min_{x \in H \neq \{y\}} \#(\theta \cap H) = 2 \).
Example: Relative vs. Absolute Tukey Depth

Minimal depth of \( x \) relative to \( y \): \[ \min_{\theta \in H \setminus \{y\}} \#(\theta \cap H) = 2. \]
Example: Relative vs. Absolute Tukey Depth

Minimal depth of \( x \) relative to \( y \): 

\[
\min_{x \in H \not\ni y} \#(\theta \cap H) = 2.
\]
Example: Relative vs. Absolute Tukey Depth

Minimal depth of $y$ relative to $x$: $\min_{y \in H \not\ni x} \#(\theta \cap H) = 1$. 
Example: Relative vs. Absolute Tukey Depth
Example: Relative vs. Absolute Tukey Depth

Upper contour set of \( y \) (in terms of relative depth) is not open!
Conclusion

Thanks for your attention!!
The Separably Convex Model

Example: All pref’s representable by $u(x) = \sum_{\ell} u^\ell(x^\ell)$ with concave $u^\ell$. 
The Separably Convex Model

Example: All pref’s representable by $u(x) = \sum_\ell u^\ell(x^\ell)$ with concave $u^\ell$.

Observation

$x$ is preferred to $y$ by all voters with top $\theta_i$ iff $x$ on shortest $L_1$-path connecting $\theta_i$ and $y$.  

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The ex-ante majority relation under separably convex preferences may have cycles between ‘distant’ alternatives ...

\[ \theta_6 = \theta_7 = (0, 0, 3) \]

\[ \theta_2 = \theta_3 = (3, 0, 0) \]

\[ \theta_4 = \theta_5 = (0, 3, 0) \]
The ex-ante majority relation under separably convex preferences may have cycles between ‘distant’ alternatives ...

\[ \theta_6 = \theta_7 = (0, 0, 3) \]

\[ \theta_2 = \theta_3 = (3, 0, 0) \]

\[ \theta_4 = \theta_5 = (0, 3, 0) \]

... but no local cycles. Indeed, we have the following result:
The $L_p$-median is the choice correspondence that selects, for all profiles $\theta$,

$$\arg\min_{x \in X} \sum_{i=1}^{n} ||\theta_i - x||_p.$$ 

For all $\theta$,

$$CW^{loc}(\theta, \Pi_{\text{sepco}}) = L_1\text{-median}.$$