Axiomatic Social Choice

Arrow's Impossibility and Beyond

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Plan for Today

Can we always design a voting rule with certain desirable properties?
Short answer (by two main theorems): No. Long answer: It depends.

- Arrow's Theorem (1951)
- Muller-Satterthwaite Theorem (1977)
- Restricted Domains

Details of the theorem proofs can be found in the following review paper:

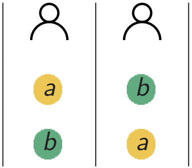
U. Endriss. Logic and Social Choice Theory. In Logic and Philosophy Today, 2011.

Warm-up

Take a finite set of voters $N = \{1, ..., n\}$ and a finite set of m alternatives A. We are working resolute voting rules:

$$F : \mathcal{L}(A)^n \to A$$

Fact: It is impossible to find a voting rule for two voters and two alternatives that is resolute, anonymous, and neutral.



For the remainder, we define $N^{\boldsymbol{P}}_{a \succ b} = \{i \in N \mid a \succ_i b \text{ in } \boldsymbol{P}\}$.

Axiom: The Pareto Principle

A voting rule F is called Paretian if, whenever all voters rank alternative a higher than alternative b, then b cannot win:

$$N^{\boldsymbol{P}}_{a \succ b} = N \textit{ implies that } F(\boldsymbol{P}) \neq b$$

Does the Borda rule satisfy the Pareto principle?

Axiom: Independence of Irrelevant Alternatives

If alternative a wins and b does not, then a is socially preferred to b.

Whether a is socially preferred to b should depend only on the relative rankings of a and b in the profile (not on other, irrelevant, alternatives).

F is called independent if for any two profiles $\boldsymbol{P}$ and $\boldsymbol{P'}$ and two alternatives $a \neq b$ such that $N^{\boldsymbol{P}}_{a \succ b} = N^{\boldsymbol{P'}}_{a \succ b}$:

$$F(\boldsymbol{P}) = a \text{ implies that } F(\boldsymbol{P'}) \neq b$$

Intuitively, the reason why b loses in $\boldsymbol{P}$ is because a is considered a better option by the group, and this reason remains in $\boldsymbol{P'}$.

Does the Borda rule satisfy IIA?

Arrow's Impossibility Theorem

F is a dictatorship if there is a voter i (the dictator) such that **for all** profiles $\boldsymbol{P} \in \mathcal{L}(A)^n$, the outcome $F(\boldsymbol{P})$ is the first alternative of $\succ_i$.

Theorem. *Any resolute voting rule for $m \geqslant 3$ that is Paretian and independent must be a dictatorship.*

- Impossibility: independence + Pareto + nondictatoriality
- Characterisation: dictatorship = independence + Pareto

K. Arrow. Social Choice and Individual Values. John Wiley and Sons, 1951.

Proof Sketch

- Fix a voting rule F that is Paretian and independent.
- Call a coalition $C \subseteq N$ decisive for $(a, b) \in A^2$ if:

$$a \succ_i b \text{ for all } i \in C \text{ implies that } F(\boldsymbol{P}) \neq b$$

- Contagion Lemma: If C with $|C| \geqslant 2$ is decisive for a pair (a, b), then it is decisive for all pairs $(x, y) \in A^2$.
- Contraction Lemma: If C with $|C| \geqslant 2$ is decisive for all pairs, then some $C' \subset C$ is decisive for all pairs too.
- By induction, there is a decisive coalition of size 1 (the dictator)!

Contagion Lemma

Claim. C is decisive for (a, b), then decisive for all $(x, y) \in A^2$.

Proof. Suppose that a, b, x, y are all distinct (other cases, similar). Construct a profile as follows:

- Voters in C: $x \succ a \succ b \succ y \succ \mathit{rest}$
- Others: $\{x \succ a,\ b \succ y, \text{ and } b \succ a\} \succ \mathit{rest}$

Decisive C for (a, b) implies that b must lose.
Pareto implies that a must lose (from x) and y must lose (from b).

$\Rightarrow$ x must win (and y must lose). By independence, y still loses when every ranking besides x vs.y changes: Note that $C \subseteq N^{\boldsymbol{P}}_{x \succ y}$.
$\Rightarrow$ For every profile $\boldsymbol{P}$ with $C \subseteq N^{\boldsymbol{P}}_{x \succ y}$, we get $y \neq F(\boldsymbol{P})$.

Contraction Lemma

Claim. If C with $|C| \geqslant 2$ is a decisive coalition on all pairs of alternatives, then so is some nonempty coalition $C' \subset C$.

Proof. Take any nonempty C_1, C_2 with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$.

Recall that $m \geqslant 3$. Take $a, b, c \in A$ and construct a profile as follows:

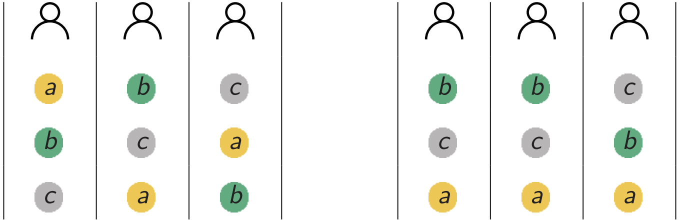
- Voters in C_1: $a \succ b \succ c \succ rest$
- Voters in C_2: $b \succ c \succ a \succ rest$
- Voters in $N \setminus C$: $c \succ a \succ b \succ rest$

$C_1 \cup C_2$ is decisive, so c cannot win (loses to b). Two cases (by Pareto):

1. The winner is a. Note that C_1 ranks $a \succ c$. By independence, in any profile where C_1 ranks $a \succ c$, c will lose (to a). So C_1 is decisive on (a, c), and so decisive on all pairs (contagion lemma). ✓
2. The winner is b, that is, a loses to b: C_2 ranks $b \succ a$, so (as above), C_2 is decisive on all pairs. ✓

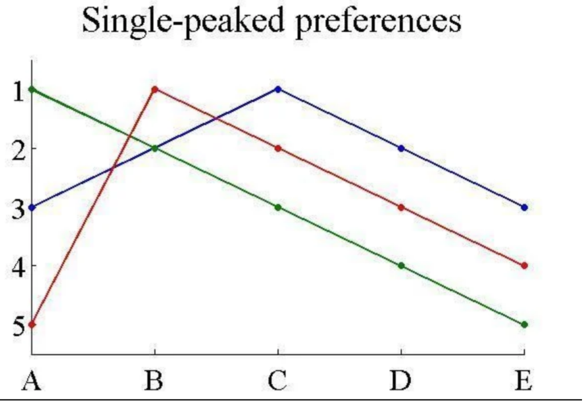
A Way Out: Removing Independence

We may want to reconsider independence. Incompatible with Condorcet:



Without independence, there are many Paretian rules (e.g., Borda).

A Way Out: Restricted Domains



For single-peaked domains, the median rule (choose the median of all individual peaks) satisfies pareto and independence.

Duncan Black. The Theory of Committees and Elections, 1958.

Strong (Maskin) Monotonicity

Decreasing the support of a loser preserves the winner.

For any alternative a and profiles $\boldsymbol{P}$ and $\boldsymbol{P'}$,

$$\text{if } N^{\boldsymbol{P}}_{a \succ b} \subseteq N^{\boldsymbol{P'}}_{a \succ b} \text{ for all } b \neq a,$$

then $F(\boldsymbol{P}) = a$ implies that $F(\boldsymbol{P'}) = a$.

The Muller-Sattertwaite Theorem

F is called surjective (or nonimposed) if for every alternative $a \in A$ there exists a profile $\boldsymbol{P}$ such that $F(\boldsymbol{P}) = x$.

Theorem. *Any resolute voting rule for $m \geqslant 3$ that is surjective and strongly monotonic is a dictatorship.*

Proof sketch. We will use Arrow's theorem, showing that:

- Strong monotonicity implies independence.
- Surjectivity and strong monotonicity imply the Pareto principle.

E. Muller and M.A. Satterthwaite. The Equivalence of Strong Positive Association and Strategy-Proofness. Journal of Economic Theory, 1977.

Deriving Independence

Recall independence : If $N^{\boldsymbol{P}}_{a \succ b} = N^{\boldsymbol{P}'}_{a \succ b}$ for $a \neq b$, then $F(\boldsymbol{P}) = a$ implies that $F(\boldsymbol{P}) \neq b$. We will show it is implied by strong monotonicity.

Take F strongly monotonic, $a \neq b$, $N^{\boldsymbol{P}}_{a \succ b} = N^{\boldsymbol{P}'}_{a \succ b}$, and $F(\boldsymbol{P}) = a$.

Construct a third profile $\boldsymbol{P}''$:

- All voters rank a and b in the top two positions.
- The relative rankings of a vs. b are as in $\boldsymbol{P}$: $N^{\boldsymbol{P}''}_{a \succ b} = N^{\boldsymbol{P}}_{a \succ b}$.

Strong monotonicity: $F(\boldsymbol{P}) = a$ implies that $F(\boldsymbol{P}'') = a$.
Strong monotonicity: $F(\boldsymbol{P}') = b$ would imply that $F(\boldsymbol{P}'') = b$.
$\Rightarrow F(\boldsymbol{P}') \neq b$. ✓

Deriving the Pareto Principle

Recall Pareto principle : If $N^{\boldsymbol{P}}_{a \succ b} = N$, then $F(\boldsymbol{P}) = b$. We will show it is implied by surjectivity and strong monotonicity.

Take F surjective and strongly monotonic (so also independent), and any two alternatives a, b.

Surjectivity: a will win for some profile $\boldsymbol{P}$.

Starting in $\boldsymbol{P}$, move a higher than b for all voters.
Monotonicity: a still wins.

Independence: b does not win in any profile where all voters rank a higher than b.

Summary

We have presented and proved two classical impossibility theorems of the axiomatic method in social choice theory. They show that certain combinations of desirable axioms mandate a dictatorship.

- Arrow: Pareto and independence
- Muller-Satterthwaite: surjectivity and strong monotonicity

→ **Next:** Strategic behaviour.