Games in Descriptive Set Theory, or: it's all fun and games until someone loses the axiom of choice

Hugo Nobrega

Cool Logic

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Presentation outline

[0]

Descriptive set theory

The real line $\mathbb{R}$ can have some pathologies (in ZFC): for example, not every set of reals is Lebesgue measurable, there *may* be sets of reals of cardinality strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$, etc.

Descriptive set theory, the theory of definable sets of real numbers, was developed in part to try to fill in the template

"No definable set of reals of complexity c can have pathology P"

Baire space $\mathbb{N}^{\mathbb{N}}$

For a lot of questions which interest set theorists, working with $\mathbb{R}$ is unnecessarily clumsy.

It is often better to work with other (Cauchy-)complete topological spaces of cardinality $|\mathbb{R}|$ which have bases of cardinality $|\mathbb{N}|$ (a.k.a. Polish spaces), and this is enough (in a technically precise way).

The Baire space $\mathbb{N}^{\mathbb{N}}$ is especially nice, as I hope to show you, and set theorists often (usually?) mean this when they say "real numbers".

The topology of $\mathbb{N}^{\mathbb{N}}$

We consider $\mathbb{N}^{\mathbb{N}}$ with the product topology of discrete $\mathbb{N}$.

...

This topology is generated by the complete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is least such that } x(n) \neq y(n). \end{cases}$$

For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we denote

$$[\sigma] := \{x \in \mathbb{N}^{\mathbb{N}} \,;\, \sigma \text{ is a prefix of } x\}$$

Then

$$\{[\sigma] \,;\, \sigma \in \mathbb{N}^{<\mathbb{N}}\}$$

is a (countable) basis for the topology of $\mathbb{N}^{\mathbb{N}}$.

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For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we denote

$$[\sigma] := \{x \in \mathbb{N}^{\mathbb{N}} \, ; \, \sigma \subset x\}$$

Then

$$\{[\sigma] \, ; \, \sigma \in \mathbb{N}^{<\mathbb{N}}\}$$

is a (countable) basis for the topology of $\mathbb{N}^{\mathbb{N}}$.

The computational flavor of $\mathbb{N}^{\mathbb{N}}$

Thus a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ is open iff there exists some $A \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$X \in \bigcup_{\sigma \in A} [\sigma].$$

Hence, if X is open and we want to decide if some given x is in X, then we can *inspect* longer and longer finite prefixes of x,

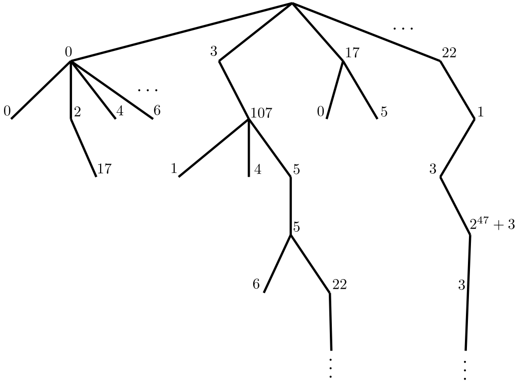
$$\begin{array}{l} \langle x_0 \rangle \\ \langle x_0, x_1 \rangle \\ \langle x_0, x_1, x_2 \rangle \\ \vdots \end{array}$$

and in case $x \in X$ is indeed true, at some finite stage we will "know" this (if $x \notin X$ then all bets are off).

This is analogous to the recursively enumerable sets in computability theory.

The combinatorial flavor of $\mathbb{N}^\mathbb{N}$

A tree is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which is closed under prefixes.



An element $x \in \mathbb{N}^\mathbb{N}$ is an infinite path of a tree T if all finite prefixes of x are in T. The body of T is the set of all its infinite paths, denoted $[T]$.

Theorem

The closed sets of $\mathbb{N}^\mathbb{N}$ are exactly the bodies of trees.

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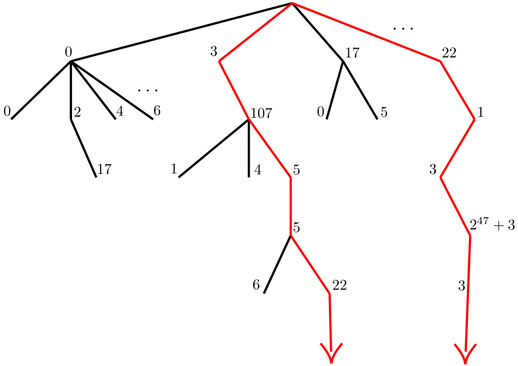

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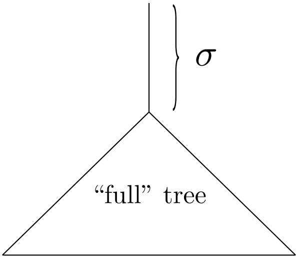
Notation clash?

We use the same notation for basic open sets, $[\sigma]$, as for bodies of trees, $[T]$.

But actually $[\sigma]$ is *also* the body of a certain tree:



Thus every basic open set is also closed, in stark contrast to $\mathbb{R}$ which has only *two* clopen sets, $\emptyset$ and $\mathbb{R}$.

The Borel hierarchy

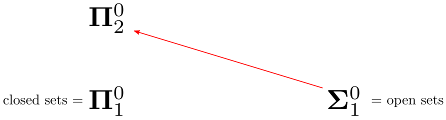
The Borel hierarchy

$\mathbf{\Sigma}^0_1$ = open sets

The Borel hierarchy

closed sets = $\mathbf{\Pi}^0_1$ $\mathbf{\Sigma}^0_1$ = open sets

The Borel hierarchy



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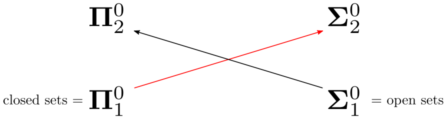

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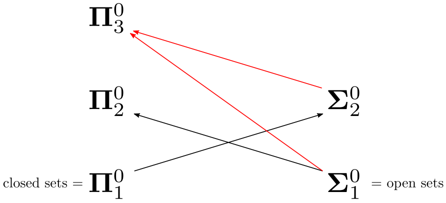

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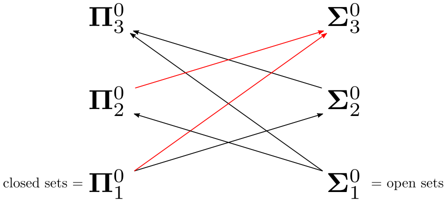

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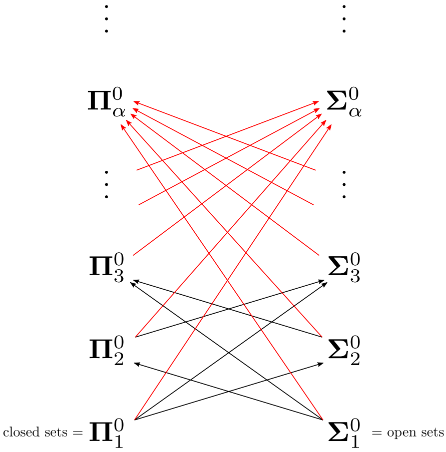

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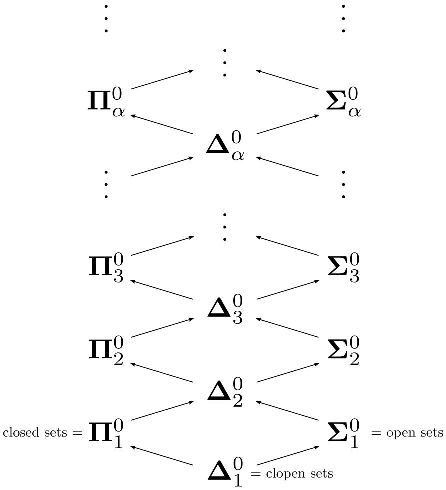

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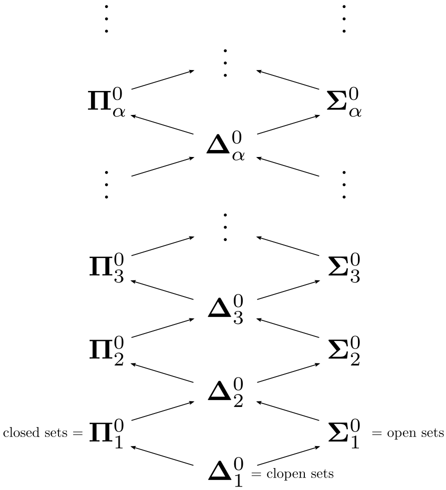
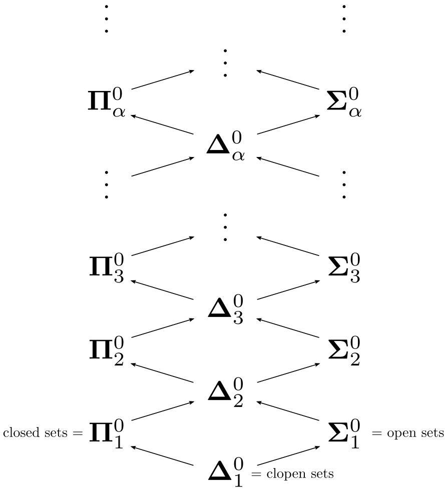


A set is Borel iff it belongs to $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$

The Borel hierarchy



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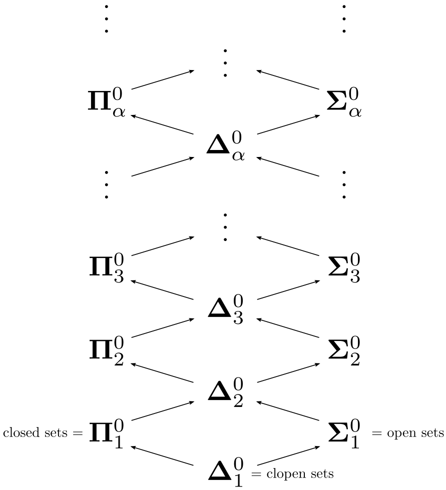

A set is Borel iff it belongs to $\bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha$

Presentation outline

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Gale-Stewart games

Given $A \subseteq \mathbb{N}^{\mathbb{N}}$, the Gale-Stewart game for A is played between two players, $\mathbf{I}$ and $\mathbf{II}$, in $\mathbb{N}$ rounds.

Player $\mathbf{I}$ plays in even rounds, $\mathbf{II}$ in odd rounds.

At round n the corresponding player picks $x_n \in \mathbb{N}$ (with perfect information).

	Round
$\mathbf{I}$	
$\mathbf{II}$	

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	Round
	0
$\mathbf{I}$	x_0
$\mathbf{II}$	

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	Round	
	0	1
$\mathbf{I}$	x_0	
$\mathbf{II}$		x_1

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	Round		
	0	1	2
$\mathbf{I}$	x_0		x_2
$\mathbf{II}$		x_1	

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	Round			
	0	1	2	3
$\mathbf{I}$	x_0		x_2	
$\mathbf{II}$		x_1		x_3

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	Round				
	0	1	2	3	4
$\mathbf{I}$	x_0		x_2		x_4
$\mathbf{II}$		x_1		x_3	

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	Round					
	0	1	2	3	4	$\cdots$
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	0	1	2	3	4	$\cdots$
$\mathbf{I}$	x_0		x_2		x_4	
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Player $\mathbf{I}$ wins iff $x = \langle x_0, x_1, x_2, \ldots \rangle \in A$, and A is determined if one of the players has a winning strategy in the game for A.

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Note that the determinacy of A is a kind of infinitary De Morgan law:

$$\neg\left[\exists x_0 \forall x_1 \exists x_2 \forall x_3 \cdots \ \langle x_0, x_1, \ldots \rangle \in A\right]$$

iff

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \cdots \ \langle x_0, x_1, \ldots \rangle \notin A.$$

The Axiom of Determinacy

In ZFC, the following is the best one can prove.

Theorem (Gale and Stewart; Martin)

Every Borel set is determined.

The Axiom of Determinacy is the statement "every subset of $\mathbb{N}^{\mathbb{N}}$ is determined".

In ZFC this is straight-up false:

Theorem (ZFC)

There exists a non-determined set.

But this uses the axiom of choice in an essential way; there is a statement ϕ involving large cardinals such that:

Theorem (Woodin)

If ZFC $+ \phi$ is consistent, then so is ZF $+$ AD.

The Axiom of Determinacy

Life in ZF + AD is very different from that in ZFC.

Theorem (ZF + AD)

1. The Continuum Hypothesis holds*;
2. every set of reals is Lebesgue measurable (likewise for many other pathologies);
3. $\aleph_1$ and $\aleph_2$ are measurable cardinals (!), but all other $\aleph_n$ have cofinality $\aleph_2$ (!!).

$$\vdots$$

We move back to the safe haven of ZFC for the rest of the talk.

Presentation outline

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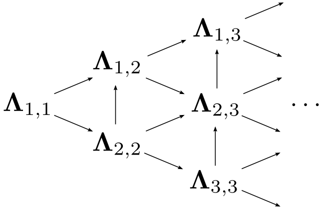
A hierarchy of functions

One way to measure the complexity of a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is by how much it deforms the Borel hierarchy (under preimages).

Hence continuous functions are "simple", but Baire class 1 functions (pointwise limits of continuous functions) are slightly more complex, and so on.

We define

$$\mathbf{\Lambda}_{\alpha,\beta} := \{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} ; \forall X \in \mathbf{\Sigma}^0_\alpha .\ f^{-1}[X] \in \mathbf{\Sigma}^0_\beta\}$$



Today we will mainly focus on the Baire classes $\mathbf{\Lambda}_{1,\alpha}$.

The general framework

In the games we will consider, players $\mathbf{I}$ (male) and $\mathbf{II}$ (female) are given a function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and again play in $\mathbb{N}$ rounds with perfect information.

However, now both $\mathbf{I}$ and $\mathbf{II}$ play at each round n:
$\mathbf{I}$ plays a natural number x_n, and $\mathbf{II}$ plays some y_n from a certain set M of moves.

Therefore in the long run they build $x = \langle x_0, x_1, \dots \rangle \in \mathbb{N}^\mathbb{N}$ and $y = \langle y_0, y_1, \dots \rangle \in M^\mathbb{N}$, respectively.

There is a set $R \subseteq M^\mathbb{N}$ of rules, and $\mathbf{II}$ loses if $y \notin R$.

There is an interpretation function $i : R \to \mathbb{N}^\mathbb{N}$, and player $\mathbf{II}$ wins a run of the game iff $y \in R$ and $i(y) = f(x)$.

We say that a game characterizes a class $\mathcal{C}$ of functions if

A function f is in $\mathcal{C}$
iff
Player $\mathbf{II}$ has a winning strategy in the game for f.

The Wadge game

In the Wadge game for f, player $\mathbf{II}$'s moves are

- play a natural number; or
- pass.

The rule is that she must play natural numbers infinitely often.

Theorem (Wadge (Duparc?))

The Wadge game characterizes the continuous functions (i.e., $\mathbf{\Lambda}_{1,1}$).

The eraser game

In the eraser game for f, player $\mathbf{II}$'s moves are

- play a natural number;
- pass; or
- erase a past move.

The rules are that she must

- play natural numbers infinitely often; and
- only erase each position of her sequence finitely many times.

Theorem (Duparc)

The eraser game characterizes $\mathbf{\Lambda}_{1,2}$.

The backtrack game

In the backtrack game for f, player $\mathbf{II}$'s moves are

- play a natural number;
- pass; or
- start over from scratch (backtrack).

The rules are that she must

- play natural numbers infinitely often; and
- backtrack finitely many times.

Theorem (Andretta)

The backtrack game characterizes $\mathbf{\Lambda}_{2,2}$.

The tree game

In his PhD thesis at the ILLC, Brian Semmes introduced the tree game.

At round n, player $\mathbf{II}$ plays a finite tree T_n and a function $\phi_n : T_n \to \mathbb{N}$ (called a labelling)

The rules are

- For all n we must have $T_n \subseteq T_{n+1}$ and $\phi_n \subseteq \phi_{n+1}$; and
- $T := \bigcup_n T_n$ must be an infinite tree with a unique infinite path.

The interpretation function is "the labels along the infinite path of T".

Theorem (Semmes)

The tree game characterizes the Borel functions, i.e.,
those for which the preimage of any Borel set is a Borel set.

A new template

Note that each Baire class $\mathbf{\Lambda}_{1,\alpha}$ is a subset of the Borel functions.

Problem

Given $\alpha < \omega_1$, find a property Φ_α of trees such that adding

$$T \text{ must have property } \Phi_\alpha$$

as a rule to the tree game results in a game which characterizes $\mathbf{\Lambda}_{1,\alpha}$.

Examples

1. Φ_1 is "T is linear" (i.e. each node has exactly one immediate child).
2. Φ_2 is "T is finitely branching".
3. (Semmes) Φ_3 is "T is finitely branching outside of its infinite path".

Games for finite Baire classes

Given a tree T, define its pruning derivative by

$$T' := \{\sigma \in T\,;\text{ the subtree of } T \text{ rooted at } \sigma \text{ has infinite height}\}.$$

Theorem (N.)

- Φ_{2n+1} is "$T^{(n)}$ is linear"; and
- Φ_{2n+2} is "$T^{(n)}$ is finitely branching".

Interestingly, this gives a different Φ_3 than the one found by Semmes.

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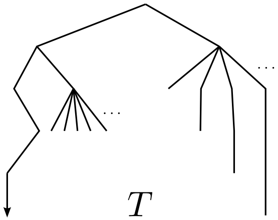

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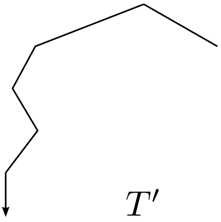

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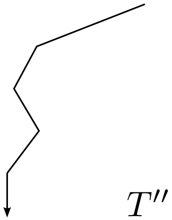

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Infinite Baire classes?

We can extend $T^{(\alpha)}$ into the transfinite by defining

$$\begin{aligned} T^{(0)} &:= T \\ T^{(\alpha+1)} &:= (T^{(\alpha)})' \\ T^{(\lambda)} &:= \bigcap_{\alpha<\lambda} T^{(\alpha)} \qquad \text{for limit } \lambda. \end{aligned}$$

Conjecture

For any limit $\lambda < \omega_1$,

- $\Phi_{\lambda+2n+1}$ is "$T^{(\lambda+n)}$ is linear".
- $\Phi_{\lambda+2n+2}$ is "$T^{(\lambda+n)}$ is finitely branching".

Full disclosure

Game characterizations of all Baire classes $\mathbf{\Lambda}_{1,\alpha}$ have independently been found by Alain Louveau, who was building on/working with Semmes after the latter's PhD.

These results have never been published.

Thanks for your attention!
Questions?