A generalization of de Vries duality to closed relations between compact Hausdorff spaces

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Stone's representation theorem for boolean algebras

The theory of boolean algebras was created in 1847 by Boole as a calculus suitable for a mathematical analysis of logic.

Properties of boolean algebras (such as the de Morgan laws) resemble set-theoretic properties (involving unions, intersections, complements).

Given a set X, the power set $\mathcal{P}(X)$ is a boolean algebra:

$$A \vee B = A \cup B, \quad A \wedge B = A \cap B, \quad 0 = \varnothing, \quad 1 = X, \quad \neg A = X \setminus A.$$

Moreover, every set of subsets of X that is closed under finite unions, finite intersections and complements ($=$ a field of subsets of X) is a boolean algebra.

Stone's representation theorem for boolean algebras [Stone, 1936]

Every boolean algebra is isomorphic to a field of subsets of some set.

Different fields of subsets may give isomorphic boolean algebras.

E.g. consider the sets $X = \{x\}$ and $Y = \{y_1, y_2\}$, equipped with the fields of subsets $A_1 := \{\varnothing, X\}$ and $A_2 := \{\varnothing, Y\}$. The boolean algebras A_1 and A_2 are isomorphic, but X and Y do not even have the same cardinality.

In order to have a bijective correspondence between boolean algebras and certain fields of sets, Stone considers what are now called Stone spaces (or Boolean spaces or profinite spaces): compact Hausdorff zero-dimensional (= the topology is generated by the closed open subsets) spaces.

E.g.: any finite discrete space, the one-point compactification of a discrete space, any arbitrary product of finite discrete spaces...

Given a Stone space X, the set $\mathrm{Clop}(X)$ of clopens (= closed open subsets) of X is a boolean algebra. Every boolean algebra is isomorphic to $\mathrm{Clop}(X)$ for some Stone space X. This assignment gives a bijective correspondence between isomorphism classes of boolean algebras and homeomorphism classes of Stone spaces [Stone, 1936].

Given a continuous function $f\colon X \to Y$ between Stone spaces,

$$\begin{aligned} f^{-1}[-]\colon \operatorname{Clop}(Y) &\longrightarrow \operatorname{Clop}(X) \\ V &\longmapsto f^{-1}[V] \end{aligned}$$

is a boolean homomorphism, and every boolean homomorphism $\operatorname{Clop}(Y) \to \operatorname{Clop}(X)$ is obtained in this way.

The assignment $f \mapsto f^{-1}[-]$ is a bijection between the set of continuous functions $X \to Y$ and the set of boolean homomorphisms $\operatorname{Clop}(Y) \to \operatorname{Clop}(X)$.

The bijective correspondence between boolean algebras and Stone spaces, together with their appropriate maps, can be expressed via category theory as a duality between the category **Stone** of Stone spaces and continuous maps and the category **BA** of boolean algebras and boolean homomorphisms. The word duality refers to the fact that the direction of morphisms is inverted.

De Vries duality

Stone duality for boolean algebras used topology to do some algebra of established interest.

Mathematicians started wondering whether we can use algebra to do topology, or at least make some interesting algebra out of some topology of established interest.

De Groot suggested his student de Vries to obtain a Stone-like duality for the category **KHaus** of compact Hausdorff spaces and continuous functions. De Vries started working on it, and he realized a connection with compactifications. This ended up in de Vries' PhD thesis "Compact spaces and compactifications: an algebraic approach" (1962).

A regular open subset of a space X is a subset U of X such that $U = \mathrm{int}(\mathrm{cl}(U))$ (in particular, it is open).

Example

1. $U \coloneqq [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ is a regular open subset of $[0, 1]$, because
$$\mathrm{cl}(U) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$
whose interior is U.
2. $U \coloneqq \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$ is not a regular open subset of $[0, 1]$, because
$$\mathrm{cl}(U) = [0, 1],$$
whose interior is $[0, 1]$, which is not U.

For every topological space X, the set $\mathrm{RO}(X)$ of regular open subsets is a complete boolean algebra with respect to the inclusion order [MacNeille, 1937], [Tarski, 1937].

$$\begin{aligned} A \vee B &= \mathrm{int}(\mathrm{cl}(A \cup B)); \\ A \wedge B &= A \cap B; \\ 0 &= \emptyset; \\ 1 &= X; \\ \neg A &= \mathrm{int}(X \setminus A). \end{aligned}$$

To a compact Hausdorff space X, de Vries associated the boolean algebra $\mathrm{RO}(X)$ of regular open subsets of X, equipped with a binary relation $\prec$ (well inside) on $\mathrm{RO}(X)$ defined by

$$A \prec B \iff \mathrm{cl}(A) \subseteq B.$$

Example

In $[0,1]$, we have $\left[0,\frac{1}{3}\right) \prec \left[0,\frac{2}{3}\right)$ and $\left[0,\frac{1}{3}\right) \not\prec \left[0,\frac{1}{3}\right) \cup \left(\frac{2}{3},1\right]$.

Up to homeomorphism, X is completely determined by the structure of boolean algebra of $\mathrm{RO}(X)$ together with the relation $\prec$: two spaces X and Y are homeomorphic iff there is a boolean isomorphism between $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ that preserves and reflects $\prec$.

Definition

A de Vries algebra is a complete boolean algebra equipped with a binary relation $\prec$ s.t.:

1. $a \prec 1$;
2. $(a \prec b,\ a \prec c)$ implies $a \prec b \wedge c$;
3. $a \prec b$ implies $\neg b \prec \neg a$;
4. $a \prec b$ implies $a \leq b$;
5. $a \leq b \prec c \leq d$ implies $a \prec d$;
6. $a \prec b$ implies that there exists c such that $a \prec c \prec b$.
7. $a \neq 0$ implies that there exists $b \neq 0$ such that $b \prec a$.

For every compact Hausdorff space X, $\mathrm{RO}(X)$ is a de Vries algebra. Every de Vries algebra $(A, \prec)$ is isomorphic to $\mathrm{RO}(X)$ for some compact Hausdorff space X (unique up to homeomorphism) [de Vries, 1962].

This gives a one-to-one correspondence between homeomorphism classes of compact Hausdorff spaces and isomorphism classes of de Vries algebras.

To a continuous function $f\colon X \to Y$ between compact Hausdorff spaces, de Vries associates the function

$$\begin{aligned} f^*\colon \mathrm{RO}(Y) &\longrightarrow \mathrm{RO}(X) \\ V &\longmapsto \mathrm{int}(\mathrm{cl}(f^{-1}[V])). \end{aligned}$$

A function $g\colon A \to B$ between de Vries algebras is called chary (or a de Vries morphism) if

1. $g(0) = 0$;
2. $g(a \wedge b) = g(a) \wedge g(b)$;
3. $a \prec b$ implies $\neg g(\neg a) \prec g(b)$;
4. $g(a) = \bigvee \{g(b) \mid b \prec a\}$.

The function $f^*\colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$ above is chary. The assignment $f \mapsto f^*$ is a bijection between the set of continuous functions from X to Y and the set of chary functions from $\mathrm{RO}(Y)$ to $\mathrm{RO}(X)$.

A chary function might fail to be a boolean homomorphism.

Given two continuous functions $X \xrightarrow{f} Y \xrightarrow{g} Z$, the functions $(g \circ f)^*$ and $f^* \circ g^*$ might differ in general.

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad X \xrightarrow{g \circ f} Z$$

$$\mathrm{RO}(X) \xleftarrow{f^*} \mathrm{RO}(Y) \xleftarrow{g^*} \mathrm{RO}(Z), \qquad \mathrm{RO}(X) \xleftarrow{(g \circ f)^*} \mathrm{RO}(Z)$$

Composition of chary functions might fail to be chary.

Definition

We let **DeV** denote the category whose objects are de Vries algebras and whose morphisms are the chary functions, with composition of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ given by the function $g * f \colon A \to C$ given by

$$(g * f)(a) = \bigvee \{g(f(b)) \mid b \prec a\}.$$

Theorem ([de Vries, 1962])

The category **KHaus** *of compact Hausdorff spaces and continuous functions is dually equivalent to the category* **DeV** *of de Vries algebras and chary functions.*

De Vries algebras have connections with the theory of compactifications and the theory of proximity spaces.

Alternative to the morphisms

The fact that the composition of morphisms is not function composition is a major drawback of de Vries duality, and it might be seen as an indication of the fact that chary functions are not the most natural way to encode continuous functions.

To remedy this, we propose to work with certain relations instead of chary functions.

Working with relations instead of functions is not a new thing.

- [Abramsky, Jung, 1994]: the category of spectral spaces and continuous functions is dually equivalent to the category of bounded distributive lattices and certain relations as morphisms, with composition given by relation composition.
- [Jung, Sünderhauf, 1996]: the category of so-called stably compact spaces (a generalization of compact Hausdorff spaces) is equivalent to the category of so-called strong proximity lattices and certain relations as morphisms, with composition given by relation composition.

Given a continuous function $f: X \to Y$ between compact Hausdorff spaces, we define a relation $S_f\colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$, as follows:

$$U \; S_f \; V \iff \mathrm{cl}[U] \subseteq f^{-1}[V] \iff f[\mathrm{cl}(U)] \subseteq V.$$

For example, if $f: X \to X$ is the identity, $S_f = \prec$.

As usual, given relations $X \xrightarrow{R} Y \xrightarrow{S} Z$, their composite $S \circ R\colon X \to Z$ (or $R; S$) is defined by

$$x \, (S \circ R) \, z \iff \exists y \in Y \text{ s.t. } x \, R \, y \, S \, z.$$

We have $S_{g \circ f} = S_g \circ S_f$, which means that taking relation composition as morphisms (in a category yet to be defined) will work.

Definition

A relation $S\colon A \to B$ between de Vries algebras is called a functional compatible subordination relation if

1. S is a subordination relation from the boolean algebra A to the boolean algebra B (cf. [Bezh., Bezh., Sour., Ven., 2017]), i.e.
   1.1 $0\ S\ b$;
   1.2 $a\ S\ 1$;
   1.3 if $a_1\ S\ b$ and $a_2\ S\ b$, then $(a_1 \vee a_2)\ S\ b$;
   1.4 if $a\ S\ b_1$ and $a\ S\ b_2$, then $a\ S\ (b_1 \wedge b_2)$;
   1.5 if $a' \leq a\ S\ b \leq b'$, then $a'\ S\ b'$;
2. S is compatible (with the relations $\prec_A$ and $\prec_B$), i.e.

$$a\ S\ b \iff \exists\ a' \in A\ \exists\ b' \in B\ \text{ s.t. } a \prec_A a'\ S\ b' \prec_B b;$$

3. S is functional, i.e.
   3.1 if $aS0$, then $a = 0$;
   3.2 if $b_1 \prec_B b_2$, then there is $a \in A$ s.t. $\neg a\ S\ \neg b_1$ and $a\ S\ b_2$.

Definition

We let $\mathbf{DeV^F}$ denote the category

- whose objects are de Vries algebras, and
- whose morphisms are functional compatible subordination relations.

Composition is usual composition of relations. The identity on an object $(A, \prec)$ is the relation $\prec: A \to A$.

Main Theorem (1/2)

The category **KHaus** of compact Hausdorff spaces and continuous functions is equivalent to the category $\mathbf{DeV^F}$ of de Vries algebras and functional compatible subordination relations.

This gives an alternative to classical de Vries duality.

Advantage: composition of morphisms is the usual composition of relations.

Definition

We let $\mathbf{KHaus^R}$ denote the category

- whose objects are compact Hausdorff spaces, and
- whose morphisms from X to Y are the closed relations $R \subseteq X \times Y$ (equivalently, those relations $R\colon X \to Y$ such that the R-image of a compact subset of X is compact and the R-preimage of a closed subset of Y is closed).

Composition of morphisms is composition of relations. The identity morphism is the equality relation.

Given a closed relation $R\colon X \to Y$ between compact Hausdorff spaces, we define a relation $S_R\colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$, as follows:

$$U \; S_R \; V \iff R[\mathrm{cl}(U)] \subseteq V.$$

Main Theorem (2/2)

The category $\mathbf{KHaus}^{\mathbf{R}}$ of compact Hausdorff spaces and closed relations between them is equivalent to the category $\mathbf{DeV}^{\mathbf{S}}$ of de Vries algebras and compatible subordination relations between them.

The equivalence between **KHaus** and $\mathbf{DeV}^{\mathbf{F}}$ is obtained from this by restricting to "functional" morphisms.

Piggyback on an equivalence for Stone spaces and closed relations

We piggyback on a generalization of Stone and Halmos duality.

Stone duality = duality for Stone spaces and continuous functions between them.

Halmos duality = duality for Stone spaces and continuous relations between them.

The generalization we need is an equivalence for Stone spaces and closed relations between them.

A more general result for bounded distributive lattices and Priestley spaces was obtained via order-enriched categories in [Jung, Kurz, Moshier, 2019].

Definition

We let $\mathbf{Stone^R}$ denote the category of Stone spaces and closed relations between them. Composition is composition of relations. The identity morphism is the equality relation.

To a Stone space X one associates the boolean algebra $\mathrm{Clop}(X)$. To a closed relation $R\colon X \to Y$ one associates the relation $S_R\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ defined by

$$U \; S_r \; V \iff R[U] \subseteq V.$$

Definition

A subordination relation $S\colon A \to B$ between boolean algebras is a relation s.t.

1. $0 \; S \; b$;
2. $a \; S \; 1$;
3. if $a_1 \; S \; b$ and $a_2 \; S \; b$, then $(a_1 \vee a_2) \; S \; b$;
4. if $a \; S \; b_1$ and $a \; S \; b_2$, then $a \; S \; (b_1 \wedge b_2)$;
5. if $a' \leq a \; S \; b \leq b'$ then $a' \; S \; b'$;

This generalizes the notion of subordination relation on a boolean algebra in [Bezh., Bezh., Sour., Ven., 2017].

Definition

We let $\mathbf{BA^S}$ denote the category of boolean algebras and subordination relations between them. Composition of morphisms is relation composition. The identity morphism on an object A is the order $\leq$.

Theorem

The categories $\mathbf{Stone}^{\mathsf{R}}$ *and* $\mathbf{BA}^{\mathsf{S}}$ *are equivalent (and also dually equivalent).*

Our equivalence between $\mathbf{KHaus}^{\mathsf{R}}$ and $\mathbf{DeV}^{\mathsf{S}}$ is a consequence (and then also the equivalence between $\mathbf{KHaus}$ and $\mathbf{DeV}^{\mathsf{F}}$ follows), as explained in the next slides.

A De Vries algebra can be seen as a pair $(A, \prec)$ where A is a boolean algebra (so, an object of $\mathbf{BA^S}$) and $\prec$ is a subordination relation from A to A (so, an endomorphism on A in $\mathbf{BA^S}$) satisfying additional conditions; for example, it is idempotent: $\prec \circ \prec = \prec$.

Definition ([Freyd, 1964])

The Karoubi envelope (or splitting by idempotents or Cauchy completion) of a category **C** is the category **K**(**C**)

- whose objects are pairs (X, f), where $X \in \mathbf{C}$ and f is an endomorphism of X such that $f \circ f = f$, and
- whose morphisms from (X_1, f_1) to (X_2, f_2) are the morphisms $g \colon X_1 \to X_2$ in C such that $f_2 \circ g = g = g \circ f_1$.

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ {\scriptstyle f_1}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[g]{} & X_2 \end{array}$$

Composition is composition in **C**. The identity on (X, f) is f.

Every de Vries algebra is an object of $\mathbf{K}(\mathbf{BA}^{\mathbf{S}})$. A morphism $(A, \prec) \to (B, \prec)$ in $\mathbf{K}(\mathbf{BA}^{\mathbf{S}})$ between de Vries algebras is a compatible subordination relation $S \colon A \to B$.

$$\mathbf{Stone}^{\mathbf{R}} \xleftrightarrow{\text{equiv.}} \mathbf{BA}^{\mathbf{S}}$$

$$\begin{array}{ccccc}
 & & \mathbf{K}(\mathbf{Stone}^{\mathbf{R}}) & \xleftrightarrow{\text{equiv.}} & \mathbf{K}(\mathbf{BA}^{\mathbf{S}}) \\
 & & \text{full} \big\uparrow & & \text{full} \big\uparrow \\
\mathbf{KHaus}^{\mathbf{R}} & \xleftrightarrow{\text{equiv.}} & \mathbf{Gle}^{\mathbf{R}} & \xleftrightarrow{\text{equiv.}} & \mathbf{DeV}^{\mathbf{S}} \\
\text{wide} \big\uparrow & & \text{wide} \big\uparrow & & \text{wide} \big\uparrow \\
\mathbf{KHaus} & \xleftrightarrow{\text{equiv.}} & \mathbf{Gle} & \xleftrightarrow{\text{equiv.}} & \mathbf{DeV}^{\mathbf{F}}.
\end{array}$$

A Gleason space [Bezh., Bezh., Sour., Ven., 2017] is a pair (X, E) with X a Stone space and E a closed equivalence relation on X s.t. $X \to X/E$ is a Gleason cover of X/E. Gleason spaces are objects of $\mathbf{K}(\mathbf{Stone}^{\mathbf{R}})$.

$\mathbf{Gle}^{\mathbf{R}}$:= category of Gleason spaces and “compatible” closed relations [Bezh., Gab., Hard., Jibl., 2019]. $\mathbf{Gle}^{\mathbf{R}}$ is equivalent to $\mathbf{KHaus}^{\mathbf{R}}$ (mapping (X, E) to X/E).

A similar usage of Karoubi envelopes in the context of stably compact spaces was mentioned in [Kegelmann, 2002] (and suggested by P. Taylor) and employed in [van Gool, 2012].

Conclusions

From the equivalence between $\mathbf{Stone}^{\mathbf{R}}$ and $\mathbf{BA}^{\mathbf{S}}$ one deduces an equivalence between $\mathbf{KHaus}^{\mathbf{R}}$ (compact Hausdorff spaces and closed relations) and $\mathbf{DeV}^{\mathbf{S}}$. Then one throws away some morphisms to obtain an equivalence between **KHaus** (compact Hausdorff spaces and continuous functions) and $\mathbf{DeV}^{\mathbf{F}}$.

This gives an alternative to de Vries duality, where morphisms between de Vries algebras are certain relations.

The advantage is that composition of morphisms is relation composition. Moreover, the relational approach allows the equivalence to work for $\mathbf{KHaus}^{\mathbf{R}}$ as well.

Thank you.

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