THE GOLDBLATT TRANSLATION BETWEEN ORTHOLOGIC AND KTB, REVISITED

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Summary

1. Translations into Classical logic;
2. Orthologic, Ortholattices, Orthospaces.
3. The Goldblatt Translation.
4. Properties of this translation.
5. Some further thoughts and developments.
Classical Translations
Double Negation Translation

Figure 1: Kurt Gödel (1906-1978); Valery Glivenko (1897-1940)

Definition
Given $\phi \in \mathcal{L}_{CPC}$ we define the double negation translation into $\mathcal{L}_{IPC}$, as follows:

1. $K(p) = \neg\neg p$ and $K(\bot) = \bot$;
2. $K(\phi \land \psi) = K(\phi) \land K(\psi)$;
3. $K(\neg\phi) = \neg K(\phi)$.
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Theorem (Glivenko, 1929)
For every formula \( \phi, \phi \in CPC \) if and only if \( K(\phi) \in IPC \).
Godel-McKinsey-Tarski Translation

Figure 2: Alfred Tarski (1901-1983); J.C.C. McKinsey (1908-1953)

Definition

Given $\phi \in L_{IPC}$ we define the **Godel-McKinsey-Tarski** (GMT) translation into $S_4$, as follows:

1. $GMT(p) = \Box p$ and $GMT(\bot) = \bot$;
2. $GMT(\phi \land \psi) = GMT(\phi) \land GMT(\psi)$ and $GMT(\phi \lor \psi) = GMT(\phi) \lor GMT(\psi)$;
3. $GMT(\phi \rightarrow \psi) = \Box (GMT(\phi) \rightarrow GMT(\psi))$.
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3. $\text{GMT}(\phi \rightarrow \psi) = \Box (\text{GMT}(\phi) \rightarrow \text{GMT}(\psi))$.

Theorem (Godel, 1933, McKinsey-Tarski, 1948)
For every formula $\phi \in \mathcal{L}_{IPC}$, $\phi \in IPC$ if and only if $\text{GMT}(\phi) \in S4$. 
In the case of the GMT translation much more is true:

**Definition**
Let \( L \in \text{Ext}(\text{IPC}) \) and \( M \in \text{NExt}(S4) \). We say that \( M \) is a **modal companion** of \( L \) if:

\[
\phi \in L \iff \text{GMT}(\phi) \in M.
\]

Theorem (Blok, 1976, Esakia 1976)
There is an isomorphism between the lattices \( \text{Ext}(\text{IPC}) \) and \( \text{NExt}(S4; \text{Grz}) \), mappings logics to their greatest modal companion.

This makes the GMT translation very robust, and a very useful tool for the parallel analysis of modal and intuitionistic logic. More broadly it reflects a vision of non-classical logic as modalised classical logic.
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**Figure 3:** Wim Blok (1947-2003); Leo Esakia (1934-2010)
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More broadly it reflects a vision of *non-classical logic as modalised classical logic.*
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**Slogan:** Strong translations correspond to “smooth transformations” of the classes of models of our logical systems.

**Example**
In the case of the double negation translation, given a poset $P$ seen as a Kripke frame, the corresponding Boolean model is obtained by looking at $\text{Max}(P)$.

![Figure 4: Transformation from Int. Model to Classical Model](image-url)
Example
In the case of the GMT translation, given a preordered set $P$, seen as a transitive and reflexive Kripke frame, the corresponding intuitionistic logic is obtained by taking the skeleton:

\[ \Rightarrow \]

Figure 5: Transformation from S4 model to Int. Model
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\[
\begin{align*}
\text{Figure 5: Transformation from S4 model to Int. Model}
\end{align*}
\]

Intuition: identifying points in a cluster alters only some local properties; erasing worlds destroys global properties.
Orthologic, Ortholattices, Orthospaces
Ortholattices

Definition
An algebra $O = (O, \land, \lor, \bot, 0, 1)$ is said to be an ortholattice when $(O, \land, \lor, 0, 1)$ is a bounded lattice, and $\bot$ satisfies the following properties for every $a, b \in O$:

1. $(a \land b)\bot = a\bot \lor b\bot$ and $(a \lor b)\bot = a\bot \land b\bot$;
2. $a \land a\bot = 0$ and $a \lor a\bot = 1$
3. $(a\bot)\bot = a$.

These are the same axioms of Boolean algebras except for distributivity.
Definition
An algebra $O = (O, \land, \lor, \perp, 0, 1)$ is said to be an ortholattice when $(O, \land, \lor, 0, 1)$ is a bounded lattice, and $\perp$ satisfies the following properties for every $a, b \in O$:

1. $(a \land b) \perp = a \perp \lor b \perp$ and $(a \lor b) \perp = a \perp \land b \perp$;
2. $a \land a \perp = 0$ and $a \lor a \perp = 1$
3. $(a \perp) \perp = a$.

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Example

Figure 6: Examples of Ortholattices ($MO_3$ and Benzene)
Definition
Let $\mathcal{L}_O$ be the language of ortholattices. Let $\vdash$, be a binary consequence relation in this language. Then we say that $\vdash$ is an orthologic if it is closed under uniform substitution, and satisfies the following axioms, for all $\phi, \psi, \chi \in \mathcal{L}_O$:

1. For a finite set of formulas $\Gamma$, $\Gamma \vdash \phi$ if and only if $\bigwedge \Gamma \vdash \phi$
2. $\phi \land \psi \vdash \phi$; $\phi \land \psi \vdash \psi$
3. $\phi \vdash \bot \land \bot$; $\bot \land \bot \vdash \phi$
4. $\phi \land \neg \phi \vdash \psi$
5. If $\phi \vdash \psi$ and $\phi \vdash \chi$, then $\phi \vdash \psi \land \chi$
6. If $\phi \vdash \psi$ and $\psi \vdash \chi$ then $\phi \vdash \chi$
7. If $\phi \vdash \psi$ then $\psi \bot \vdash \phi \bot$

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1. For a finite set of formulas $\Gamma$, $\Gamma \vdash \phi$ if and only if $\wedge \Gamma \vdash \phi$
2. $\phi \land \psi \vdash \phi; \phi \land \psi \vdash \psi$
3. $\phi \vdash \phi \perp \perp; \phi \perp \perp \vdash \phi$
4. $\phi \land \neg \phi \vdash \psi$
5. If $\phi \vdash \psi$ and $\phi \vdash \chi$, then $\phi \vdash \psi \land \chi$
6. If $\phi \vdash \psi$ and $\psi \vdash \chi$ then $\phi \vdash \chi$
7. If $\phi \vdash \psi$ then $\psi \perp \vdash \phi \perp$

We denote by $O$ the minimal orthologic.

Theorem
There is a dual isomorphism between $\text{Ext}(O)$, the lattice of extensions of orthologic, and $\text{Var}(\text{Ort})$, the lattice of varieties of ortholattices.
Orthoframes and orthomodels

Definition
Let \((X, R)\) be a set equipped with a binary reflexive, symmetric relation, such that whenever \(x \neq y\), there is some \(z\) such \(xRz\) and \(\neg(yRz)\) or vice-versa. We call this an orthoframe.
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\[ U^\perp := \{ x : \forall y \in U, \neg(xRy) \} \]

for the orthogonal complement of \(U\).
Given an orthoframe \((X, \perp)\), we say that a subset \(U\) is regular if \(U^\perp^\perp = U\).
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Given an orthoframe \((X, \perp)\), we say that a subset \(U\) is regular if \(U^{\perp\perp} = U\).

Given an orthoframe \((X, \perp)\), a valuation \(V : Prop \rightarrow Reg(X)\) taking values in the regular subsets of \(X\) is called an orthomodel. We write \(M\) The Kripke semantics of orthomodels is defined as follows:

1. \(M, x \models p\) iff \(x \in V(p)\);
2. \(M, x \models \phi \land \psi\) iff \(M, x \models \phi\) and \(M, x \models \psi\);
3. \(M, x \models \phi^\perp\) iff whenever \(xRy\) then \(M, y \not\models \phi\).
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Theorem (Goldblatt, 1974)
Orthologic is sound and complete with respect to orthoframes.
The Goldblatt Translation

Figure 7: Robert Goldblatt (1949-)

\[
KTB := K \oplus \Box p \rightarrow p \oplus p \rightarrow \Box \Diamond p
\]

Definition (Goldblatt Translation)
For \( \phi \in L_O \) we define the Goldblatt translation:

1. \( G(p) = \Box \Diamond G(p) \)
2. \( G(\phi \land \psi) = G(\phi) \land G(\psi) \)
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3. $G(\phi \bot) = \Box \neg G(\phi)$

Theorem
For every formula $\phi \in \mathcal{L}_O$ we have that $\phi \in O$ if and only if $G(\phi) \rightarrow G\phi(\phi) \in KTB$. 

$$KTB := K \oplus \Box p \rightarrow p \oplus p \rightarrow \Box \Diamond p$$

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Theorem
For every formula \((\phi, \psi) \in \mathcal{L}_O\) we have that \(\phi \in \mathcal{O}\) if and only if \(G(\phi) \rightarrow G(\psi) \in \textbf{KTB}\).

Proof.
(Sketch) Given an orthomodel \((X, R, V)\), we can see it as a model of \(\textbf{KTB}\), such that 
\((X, \bot, V) \Vdash \phi\) if and only if \((X, R, V) \not\Vdash G(\phi)\).
**Theorem**

For every formula \((\phi, \psi) \in \mathcal{L}_O\) we have that \(\phi \in \mathcal{O}\) if and only if \(G(\phi) \rightarrow G(\psi) \in \text{KTB}\).

**Proof.**

(Sketch) Given an orthomodel \((X, R, V)\), we can see it as a model of \(\text{KTB}\), such that \((X, \bot, V) \models \phi\) if and only if \((X, R, V) \models G(\phi)\).

Conversely, given a model \((X, R, V)\) of \(\text{KTB}\) we can take a loop-skeleton:

\[
x \equiv y \iff \forall z (xRz \leftrightarrow yRz)
\]

Using this we form a quotient \(X^* := X/\equiv\), with a relation \([x]R[y]\) if and only if \(xRy\), and \([x] \in W(p)\) if and only if \(x \in \Box\Diamond V(p)\). And we have that \((X^*, R, W) \models \phi\) if and only if \((X, R, V) \models G(\phi)\).
Analysing the Goldblatt Translation
Despite early enthusiasm with this logic, this line of work went quiet after a while.
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In the early 2000’s, Miyazaki produced a detailed analysis of the translation, hinting at the kind of theory present in the GMT translation.
Definition
Let \( O \in \Lambda(O) \) and \( L \in \text{NExt}(\text{KTB}) \). We say that \( L \) is a KTB-companion of \( O \) if:

\[
(\phi, \psi) \in O \iff G(\phi) \rightarrow G(\psi) \in L
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Definition
Let $O \in \Lambda(O)$ and $L \in \text{NExt}(\text{KTB})$. We say that $L$ is a KTB-companion of $O$ if:

$$(\phi, \psi) \in O \iff G(\phi) \rightarrow G(\psi) \in L$$

Theorem (Miyazaki, 2004)
The following hold:

1. For each $L \in \text{NExt}(\text{KTB})$, there is a logic $O \in \Lambda(O)$ such that $L$ is the modal companion of $O$; this assignment preserves Kripke completeness, tabularity and FMP.

2. For each orthologic $O \in \Lambda(O)$ with the FMP, there is a logic $L \in \text{NExt}(\text{KTB})$ such that $L$ is the modal companion of $O$; this assignment preserves tabularity and FMP.
Miyazaki never followed up on this work. It is reasonable to ask whether one could have a theory fully analogous to the GMT case, i.e., including a Blok-Esakia-style isomorphism.
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The following seems to never have been written down:

**Theorem**

*There does not exist an isomorphism between $\text{Ext}(O)$ and any lattice of extensions of KTB.*
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**Theorem**

There does **not** exist an isomorphism between $\text{Ext}(O)$ and any lattice of extensions of $\text{KTB}$.

**Proof.**

By a classic result in the theory of ortholattices, and a result of Miyazaki, we know that the bottom of the lattices of varieties look as follows:

![Diagram](image.png)

**Figure 8:** Bottom of the lattice of varieties
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Indeed, unlike in the KTB case, to a single ortholattice there could correspond multiple orthoframes; both of the following are transformed into Benzene frames:

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**Figure 9:** Two Benzene frames
We were able to show that if this translation was strong – which is measured in **categorical terms** – then there would need to be a p-morphism between the two previous frames. This does not exist, as can be inspected.
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The formal treatment of this goes through by relating translations with **adjunctions**. The property which fails above is that the unit of the adjunction is not an isomorphism, i.e., the left adjoint is not fully faithful.
Expanding the Signature of Ortholattices
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The precise nature of the phenomenon at play is not yet clear to me. But by reverse engineering some well-known situations, I think I can make my point that there is indeed *something happening here*. 
Imagine that we had just discovered pseudocomplemented distributive lattices – bounded distributive lattices of type $D = (D, \land, \lor, \neg, 0, 1)$ where the negation satisfies the following property:

$$a \land c = 0 \iff a \leq \neg c$$
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$$a \land c = 0 \iff a \leq \neg c$$

We also found that these could be modelled using posets $(P, \leq)$, in the usual way: taking valuations $V : Prop \rightarrow Up(P)$, and the semantic clause

$$(P, V), x \models \neg \phi \iff \forall y \geq x, (P, V), y \not\models \phi$$
Imagine that we had just discovered pseudocomplemented distributive lattices – bounded distributive lattices of type $D = (D, \wedge, \vee, \neg, 0, 1)$ where the negation satisfies the following property:

$$a \wedge c = 0 \iff a \leq \neg c$$

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Being experienced modal logicians, we notice that this is very similar to the S4 modal system, and rush to translate this to that system with the following translation:

1. $T(p) = \Box p$ and $T(\bot) = \bot$ and $T(\top) = \top$;
2. $T(\phi \land \psi) = T(\phi) \land T(\psi)$ and $T(\phi \lor \psi) = T(\phi) \lor T(\psi)$;
3. $T(\neg \phi) = \Box \neg T(\phi)$.
However, we start to notice a few problems:

1. We can define the pre-linearly ordered S4-frames – they are just given by
   \[ \Box(\Box p \rightarrow q) \lor \Box(\Box q \rightarrow p) \] – but these cannot be defined because the natural notion of \( p \)-morphism does not preserve linearity!
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   notion of p-morphism does not preserve linearity!

2. You find that the natural semantic transformations – algebraic, duality-theoretic –
   appear to work in the finite case, but do not extend to the infinite case.

3. Eventually you find that there is not isomorphism between the lattices of
   pseudocomplemented distributive lattices and the extensions of any extension of
   S4 – the former is countable whilst the latter is of size continuum.

What is wrong with this picture?
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Key idea: orthologic as a logic of “mutual consistency”: \( xRy \) = states at \( x \) and \( y \) are mutually consistent.

This invites a **Kripkean implication** (already alluded to, in some form, by Dalla-Chiara).
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Key idea: orthologic as a logic of “mutual consistency”: \( xRy \) = states at \( x \) and \( y \) are mutually consistent.

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The intended meaning:

\[ a \leftrightarrow b := \text{“In all worlds that are consistent with the present, if } a \text{ holds, then } b \text{ holds.} \]

This generates a different class of structures, called in my thesis Orthoimplicative systems.
When adding this implication, and extend the translation we obtain an *extended Goldblatt translation* into a specific extension of KTB, the transformations become smooth, and much more can be said.
Adding implications to Goldblatt

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Theorem (Lazy Blok-Esakia Correspondence)
There exists a surjective homomorphism between $\text{NExt}(\text{KTB}^{\text{sob}})$ and $\text{Ext}(\text{Ort} \rightarrow )$, which witnesses a strong translation and preserves properties such as tabularity, FMP, local tabularity, amongst others, and an injective homomorphism $\Lambda(\text{Ort} \rightarrow )$ to $\Lambda(\text{KTB}^{\text{sob}})$ which preserves Kripke completeness amongst other properties.
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It is left open whether this map is an isomorphism.
Thank you!
Questions?
Some facts obtained about these structures:

1. Their theory is conservative over ortholattices.
2. Every finite ortholattice, and every (infinite-dimensional) Hilbert space, admits the structure of such an implication.
3. The proposed duality allows for great simplification in reasoning (See board).
4. It satisfies a well-defined universal property:
   \[ c \leq a \rightarrow b \iff a \leq c \perp \lor b \]
5. It allows the description of natural objects such as the centre of an orthomodular lattice; this defines a sound translation from classical logic into orthoimplicative logic.
6. Some recent connections: in the case of atomistic ortholattices, the duality we introduced restricts to a known class of graphs called stiff graphs.