The Goldblatt Translation between Orthologic and KTB, revisited

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Summary

1. Translations into Classical logic;
2. Orthologic, Ortholattices, Orthospaces.
3. The Goldblatt Translation.
4. Properties of this translation.
5. Some further thoughts and developments.

Classical Translations

Double Negation Translation

Figure 1: Kurt Gödel (1906-1978); Valery Glivenko (1897-1940)

Definition
Given $\phi \in \mathcal{L}_{CPC}$ we define the double negation translation into $\mathcal{L}_{IPC}$, as follows:

1. $K(p) = \neg\neg p$ and $K(\bot) = \bot$;
2. $K(\phi \wedge \psi) = K(\phi) \wedge K(\psi)$;
3. $K(\neg\phi) = \neg K(\phi)$.

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Theorem (Glivenko,1929)
For every formula ϕ, $\phi \in \mathrm{CPC}$ if and only if $K(\phi) \in \mathrm{IPC}$.

Godel-McKinsey-Tarski Translation

Figure 2: Alfred Tarski (1901-1983); J.C.C. McKinsey (1908-1953)

Definition
Given $\phi \in \mathcal{L}_{IPC}$ we define the Godel-McKinsey-Tarski (GMT) translation into $S4$, as follows:

1. $GMT(p) = \Box p$ and $GMT(\bot) = \bot$;
2. $GMT(\phi \wedge \psi) = GMT(\phi) \wedge GMT(\psi)$ and $GMT(\phi \vee \psi) = GMT(\phi) \vee GMT(\psi)$;
3. $GMT(\phi \rightarrow \psi) = \Box(GMT(\phi) \rightarrow GMT(\psi))$.


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3. $GMT(\phi \to \psi) = \Box(GMT(\phi) \to GMT(\psi))$.

Theorem (Godel,1933, McKinsey-Tarski, 1948)
For every formula $\phi \in \mathcal{L}_{IPC}$, $\phi \in IPC$ if and only if $GMT(\phi) \in S4$.

In the case of the GMT translation much more is true:

Definition
Let $L \in \text{Ext}(\text{IPC})$ and $M \in \text{NExt}(\text{S4})$. We say that M is a modal companion of L if:

$$\phi \in L \iff GMT(\phi) \in M.$$

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Theorem (Blok, 1976, Esakia 1976)
There is an isomorphism between the lattices **Ext**(*IPC*) *and* **NExt(S4.Grz)**, *mappings logics to their greatest modal companion.*

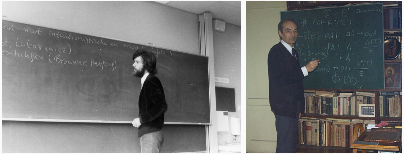

Figure 3: Wim Blok (1947-2003); Leo Esakia (1934-2010)

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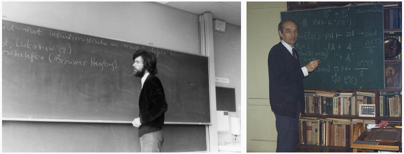

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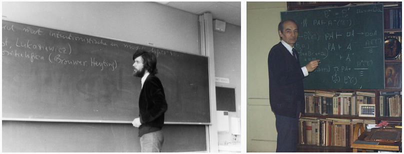

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More broadly it reflects a vision of non-classical logic as modalised classical logic.

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Example

In the case of the double negation translation, given a poset P seen as a Kripke frame, the corresponding Boolean model is obtained by looking at $\mathrm{Max}(P)$.

Figure 4: Transformation from Int. Model to Classical Model

Strong Translations (cont.)

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In the case of the GMT translation, given a preordered set P, seen as a transitive and reflexive Kripke frame, the corresponding intuitionistic logic is obtained by taking the *skeleton*:

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Figure 5: Transformation from S4 model to Int. Model

Intuition: identifying points in a cluster alters only some local properties; erasing worlds destroys global properties.

Orthologic, Ortholattices, Orthospaces

Definition

An algebra $\mathbf{O} = (O, \wedge, \vee, ^{\perp}, 0, 1)$ is said to be an ortholattice when $(O, \wedge, \vee, 0, 1)$ is a bounded lattice, and $^{\perp}$ satisfies the following properties for every $a, b \in O$:

1. $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$ and $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$;
2. $a \wedge a^{\perp} = 0$ and $a \vee a^{\perp} = 1$
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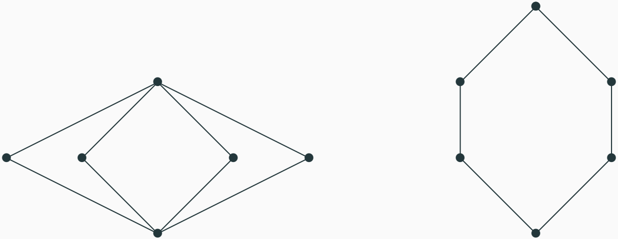

Figure 6: Examples of Ortholattices (MO_3 and Benzene)

Definition

Let $\mathcal{L}_O$ be the language of ortholattices. Let $\vdash$, be a binary consequence relation in this language. Then we say that $\vdash$ is an *orthologic* if it is closed under uniform substitution, and satisfies the following axioms, for all $\phi, \psi, \chi \in \mathcal{L}_O$:

1. For a finite set of formulas Γ, $\Gamma \vdash \phi$ if and only if $\bigwedge \Gamma \vdash \phi$
2. $\phi \wedge \psi \vdash \phi$; $\phi \wedge \psi \vdash \psi$
3. $\phi \vdash \phi^{\perp\perp}$; $\phi^{\perp\perp} \vdash \phi$
4. $\phi \wedge \neg\phi \vdash \psi$
5. If $\phi \vdash \psi$ and $\phi \vdash \chi$, then $\phi \vdash \psi \wedge \chi$
6. If $\phi \vdash \psi$ and $\psi \vdash \chi$ then $\phi \vdash \chi$
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Theorem

There is a dual isomorphism between Ext(O), *the lattice of extensions of orthologic, and* Var(Ort), *the lattice of varieties of ortholattices.*

Orthoframes and orthomodels

Definition
Let (X, R) be a set equipped with a binary reflexive, symmetric relation, such that whenever $x \neq y$, there is some z such xRz and $\neg(yRz)$ or vice-versa. We call this an orthoframe.

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We write

$$U^{\perp} := \{x : \forall y \in U, \neg(xRy)\}$$

for the orthogonal complement of U.
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Given an orthoframe $(X, \perp)$, a valuation $V : Prop \to Reg(X)$ taking values in the regular subsets of X is called an *orthomodel*. We write $\mathfrak{M}$ The Kripke semantics of orthomodels is defined as follows:

1. $\mathfrak{M}, x \Vdash p$ iff $x \in V(p)$;
2. $\mathfrak{M}, x \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, x \Vdash \phi$ and $\mathfrak{M}, x \Vdash \psi$;
3. $\mathfrak{M}, x \Vdash \phi^{\perp}$ iff whenever xRy then $\mathfrak{M}, y \nVdash \phi$.

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Theorem (Goldblatt, 1974)

Orthologic is sound and complete with respect to orthoframes.

Figure 7: Robert Goldblatt (1949-)

$$\mathsf{KTB} := \mathsf{K} \oplus \Box p \to p \oplus p \to \Box\Diamond p$$

Definition (Goldblatt Translation)
For $\phi \in \mathcal{L}_O$ we define the Goldblatt translation:

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For every formula $(\phi, \psi) \in \mathcal{L}_O$ *we have that* $\phi \in \mathcal{O}$ *if and only if* $G(\phi) \to G(\psi) \in \mathsf{KTB}$.

Proof.
(Sketch) Given an orthomodel (X, R, V), we can see it as a model of KTB, such that $(X, \perp, V) \Vdash \phi$ if and only if $(X, R, V) \Vdash G(\phi)$.

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(Sketch) Given an orthomodel (X, R, V), we can see it as a model of KTB, such that $(X, \perp, V) \Vdash \phi$ if and only if $(X, R, V) \Vdash G(\phi)$.
Conversely, given a model (X, R, V) of KTB we can take a *loop-skeleton*:

$$x \equiv y \iff \forall z(xRz \leftrightarrow yRz)$$

Using this we form a quotient $X^* := X/\equiv$, with a relation $[x]R[y]$ if and only if xRy, and $[x] \in W(p)$ if and only if $x \in \Box\Diamond V(p)$. And we have that $(X^*, R, W) \Vdash \phi$ if and only if $(X, R, V) \Vdash G(\phi)$. □

Analysing the Goldblatt Translation

A Disillusionment

Despite early enthusiasm with this logic, this line of work went quiet after a while.

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In the early 2000's, Miyazaki produced a detailed analysis of the translation, hinting at the kind of theory present in the GMT translation.

Modal Companions

Definition

Let $O \in \Lambda(\mathrm{O})$ and $L \in \mathbf{NExt}(\mathrm{KTB})$. We say that L is a KTB-companion of O if:

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Theorem (Miyazaki, 2004)
The following hold:

1. *For each* $L \in \mathbf{NExt}(\mathrm{KTB})$*, there is a logic* $O \in \Lambda(\mathrm{O})$ *such that* L *is the modal companion of* O*; this assignment preserves Kripke completeness, tabularity and FMP.*
2. *For each orthologic* $O \in \Lambda(\mathrm{O})$ *with the FMP, there is a logic* $L \in \mathbf{NExt}(\mathrm{KTB})$ *such that* L *is the modal companion of* O*; this assignment preserves tabularity and FMP.*

In Search of a Blok-Esakia

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Proof.
By a classic result in the theory of ortholattices, and a result of Miyazaki, we know that the bottom of the lattices of varieties look as follows:

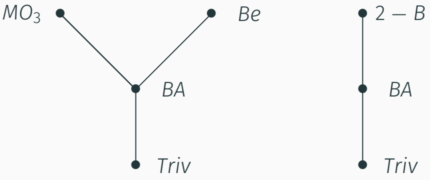

Figure 8: Bottom of the lattice of varieties

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Indeed, unlike in the KTB case, to a single ortholattice there could correspond multiple orthoframes; both of the following are transformed into Benzene frames:

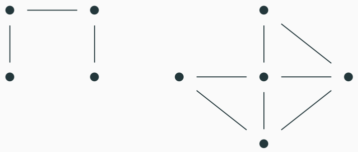

Figure 9: Two Benzene frames

Dashing Hopes (cont'd.)

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The formal treatment of this goes through by relating translations with adjunctions. The property which fails above is that the unit of the adjunction is not an isomorphism, i.e., the left adjoint is not fully faithful.

Expanding the Signature of Ortho-lattices

Problems of Duality

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On the other hand, by adding specific implications outside the signature the situation can be made better.

The precise nature of the phenomenon at play is not yet clear to me. But by reverse engineering some well-known situations, I think I can make my point that there is indeed *something happening here.*

Inventing Pseudocomplemented Distributive Lattices

Imagine that we had just discovered pseudocomplemented distributive lattices – bounded distributive lattices of type $\mathsf{D} = (D, \wedge, \vee, \neg, 0, 1)$ where the negation satisfies the following property:

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We also found that these could be modelled using posets $(P, \leq)$, in the usual way: taking valuations $V : Prop \to Up(P)$, and the semantic clause

$$(P, V), x \Vdash \neg\phi \iff \forall y \geq x, (P, V), y \nVdash \phi$$

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Being experienced modal logicians, we notice that this is very similar to the S4 modal system, and rush to translate this to that system with the following translation:

1. $T(p) = \Box p$ and $T(\bot) = \bot$ and $T(\top) = \top$;
2. $T(\phi \wedge \psi) = T(\phi) \wedge T(\psi)$ and $T(\phi \vee \psi) = T(\phi) \vee T(\psi)$;
3. $T(\neg\phi) = \Box\neg T(\phi)$.

Inventing Pseudocomplemented Distributive lattices (cont'd)

However, we start to notice a few problems:

1. We can define the pre-linearly ordered S4-frames – they are just given by $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ – but these cannot be defined because the natural notion of p-morphism does not preserve linearity!

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2. You find that the natural semantic transformations – algebraic, duality-theoretic – appear to work in the finite case, but do not extend to the infinite case.
3. Eventually you find that there is not isomorphism between the lattices of pseudocomplemented distributive lattices and the extensions of any extension of S4 – the former is countable whilst the latter is of size continuum.

What is wrong with this picture?

The Role of Implication

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Key idea: orthologic as a logic of "mutual consistency": xRy = states at x and y are mutually consistent.

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The intended meaning:

$a \hookrightarrow b :=$ " In all worlds that are consistent with the present, if a holds, then b holds.

This generates a different class of structures, called in my thesis *Orthoimplicative systems*.

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When adding this implication, and extend the translation we obtain an **extended Goldblatt translation** into a specific extension of KTB, the transformations become smooth, and much more can be said.

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Theorem (Lazy Blok-Esakia Correspondence)

There exists a surjective homomorphism between $\mathrm{NExt}(\mathsf{KTB}^{sob})$ *and* $\mathrm{Ext}(Ort^{\rightarrow})$*, which witnesses a strong translation and preserves properties such as tabularity, FMP, local tabularity, amongst others, and an injective homomorphism* $\Lambda(Ort^{\rightarrow})$ *to* $\Lambda(\mathsf{KTB}^{sob})$ *which preserves Kripke completeness amongst other properties.*

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It is left open whether this map is an isomorphism.

Thank you!

Questions?

Some facts obtained about these structures:

1. Their theory is conservative over ortholattices.
2. Every finite ortholattice, and every (infinite-dimensional) Hilbert space, admits the structure of such an implication.
3. The proposed duality allows for great simplification in reasoning (See board).
4. It satisfies a well-defined universal property:

$$c \leq a \to b \iff a \leq c^{\perp} \vee b$$

5. It allows the description of natural objects such as the centre of an orthomodular lattice; this defines a sound translation from classical logic into orthoimplicative logic.
6. Some recent connections: in the case of atomistic ortholattices, the duality we introduced restricts to a known class of graphs called stiff graphs.