Reflection algebras and conservativity spectra

Lev D. Beklemishev

LLAMA Seminar, ILLC, March 8, 2023

Hilbert's program

Consistency program:

Mathematical theories use *abstract* notions: sets, functions, spaces, etc.
Is their use logically consistent?

Conservativity program:

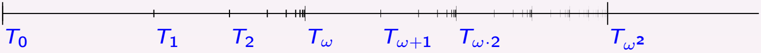
Does the use of mathematical abstractions make provable new *concrete* (combinatorial, number-theoretic) statements?

K. Gödel: Yes

If so, how can we characterize them?

Turing progressions

A.M. Turing studied the iterative process of extension of theories T by Gödelian assertions $\mathrm{Con}(T)$.



Turing progression

$$T_0 = T, \quad T_{\alpha+1} = T_\alpha + \mathrm{Con}(T_\alpha),$$

$$T_\lambda = \cup_{\alpha<\lambda} T_\alpha, \text{ for } \lambda \in \mathrm{Lim}.$$

Turing hoped to obtain a classification of all true arithmetical $\forall\exists$ statements according to the stages of this (and similar) processes – but encountered difficulties.

Turing's classification program

A.M. Turing, *System of logics based on ordinals* (1939):

> *We might also expect to obtain an interesting classification of number-theoretic theorems according to "depth". A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified.*

The problem of ordinal notations

Ordinals need to be computably represented in arithmetic, otherwise the axioms of T_α would not be r.e.

Problem

Theories T_α depend on a particular way the ordering is represented/computed rather than on its isomorphism type α.

Turing, Feferman, Kreisel:

The whole classification idea breaks down because of this problem.

Restricted Turing's program

In a restricted context Turing's approach can still work.

A.M. Turing (1939):

We can still give a certain meaning to the classification into depths with highly restricted kinds of ordinals. Suppose that we take a particular ordinal logic Λ *and a particular ordinal formula* Ψ *representing the ordinal* α *say (preferably a large one), and that we restrict ourselves to ordinal formulae of the form* $\mathrm{Inf}(\Psi, a)$.[a] *We then have a classification into depths, but the extents of all the logics which we so obtain are contained in the extent of a single logic.*

[a] These formulas define initial segments of α.

Proof-theoretic analysis by iterated reflection

An approach to Hilbert's conservativity program:

- Generalized Turing progressions T_α can be used to axiomatize theorems of a given logical complexity level (e.g. Π^0_n) of strong theories U over weak theories T, that is, to obtain
- Conservativity results of the form $\Pi^0_n(U) = T_\alpha$, for suitable ordinal notations α. Given U, n find α.
- Varying n gives us a uniform way of obtaining all the main types of proof-theoretic analysis results: consistency proofs, bounds on transfinite induction, provably recursive functions

Semilattice of Gödelian theories

Gödelian theories S

S r.e., with a fixed Σ_1 provability predicate $\Box_S$.

$\mathfrak{G}_S$ is the set of all Gödelian extensions of S mod $=_S$.

$U \leq_S T \iff S \vdash \forall x\, (\Box_T(x) \to \Box_U(x))$;

$U =_S T \iff (U \leq_S T \text{ and } T \leq_S U)$.

Then $(\mathfrak{G}_S, \wedge_S, 1_S)$ is a lower semilattice with $1_S = S$ and
$U \wedge_S T :=$ deductive closure of $U \cup T$

Reflection algebra

Reflection principles

$R_n(T)$ is an arithmetical sentence expressing *"every Σ_n-sentence provable in T is true"*.

$R_n(T)$ generalizes the consistency assertion $Con(T) = R_0(T)$.

Every formula R_n induces a monotone semi-idempotent operator $R_n : T \longmapsto S + R_n(T)$ on $\mathfrak{G}_S$.

Reflection algebra of S

is the structure $(\mathfrak{G}_S; \wedge_S, 1_S, \{R_n : n \in \omega\})$.

Semilattices with monotone operators

SLO are lower semilattices with top equipped with a family of unary operators $\mathfrak{A} = (A; \wedge, 1, \{\Diamond_i : i \in I\})$ where each $\Diamond_i$ is a monotone operator.

An operator $R : \mathfrak{A} \to \mathfrak{A}$ is:

- *monotone* if $x \leq y$ implies $R(x) \leq R(y)$;
- *semi-idempotent* if $R(R(x)) \leq R(x)$;
- *closure* if R is m., s.i. and $x \leq R(x)$.

Transfinite iterations

Def. $R : \mathfrak{G}_S \to \mathfrak{G}_S$ is *computable* if it can be defined by a computable map on the Gödel numbers of elements of $\mathfrak{G}_S$.

Suppose $(\Omega, \prec)$ is an elementary recursive well-ordering and R is a computable m.s.i. operator on $\mathfrak{G}_S$.

Theorem

There exist theories $R^\alpha(T)$ (where $\alpha \in \Omega$):
$R^0(T) =_S T$ and, if $\alpha \succ 0$,

$$R^\alpha(T) =_S \bigcup \{R(R^\beta(T)) : \beta \prec \alpha\}.$$

Each R^α is computable and m.s.i.. Under some natural additional conditions the family R^α is unique modulo provable equivalence.

Transfinite iterations

Def. $R : \mathfrak{G}_S \to \mathfrak{G}_S$ is *computable* if it can be defined by a computable map on the Gödel numbers of elements of $\mathfrak{G}_S$.

Suppose $(\Omega, \prec)$ is an elementary recursive well-ordering and R is a computable m.s.i. operator on $\mathfrak{G}_S$.

Theorem

There exist theories $R^\alpha(T)$ (where $\alpha \in \Omega$):
$R^0(T) =_S T$ and, if $\alpha \succ 0$,

$$R^\alpha(T) =_S \bigcup\{R(R^\beta(T)) : \beta \prec \alpha\}.$$

Each R^α is computable and m.s.i.. Under some natural additional conditions the family R^α is unique modulo provable equivalence.

Conservativity spectra

Let S be a Gödelian extension of EA and $(\Omega, <)$ an elementary recursive well-ordering.

- Π^0_{n+1}*-ordinal of S*, denoted $ord_n(S)$, is the sup of all $\alpha \in \Omega$ such that $S \vdash R^\alpha_n(\mathrm{EA})$;
- *Conservativity spectrum of S* is the sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ such that $\alpha_i = ord_i(S)$.

Examples of spectra:

$I\Sigma_1 : \ (\omega^\omega, \omega, 1, 0, 0, \dots)$

$\mathrm{PA} : \ (\varepsilon_0, \varepsilon_0, \varepsilon_0, \dots)$

$\mathrm{PA} + PH : \ (\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \dots)$

Conservativity spectra

Let S be a Gödelian extension of EA and $(\Omega, <)$ an elementary recursive well-ordering.

- Π^0_{n+1}*-ordinal of* S, denoted $ord_n(S)$, is the sup of all $\alpha \in \Omega$ such that $S \vdash R^\alpha_n(\mathrm{EA})$;
- *Conservativity spectrum of* S is the sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ such that $\alpha_i = ord_i(S)$.

Examples of spectra:

$I\Sigma_1 : \ (\omega^\omega, \omega, 1, 0, 0, \dots)$

$\mathrm{PA} : \ (\varepsilon_0, \varepsilon_0, \varepsilon_0, \dots)$

$\mathrm{PA} + PH : \ (\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \dots)$

Characterizing spectra

Under some naturality conditions on $(\Omega, <)$:

Theorem

A sequence $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ is a conservativity spectrum of some theory T iff $\alpha_{i+1} \leq \ell(\alpha_i)$, for all $i \in \omega$.

Here $\ell(\beta) = 0$ if $\beta = 0$, and $\ell(\beta) = \gamma$ if $\beta = \delta + \omega^\gamma$, for some δ, γ.

Remarks

- The set of all spectra $\vec{\alpha}$ such that $\forall i \in \omega\, \alpha_i < \varepsilon_0$ is the domain of the *Ignatiev model*, a well-known universal Kripke model $\mathcal{I}$ for the closed fragment of Japaridze's provability logic GLP.
- *J. Joosten (2016)* has shown the injection of $\mathcal{I}$ into the set of spectra of subtheories of PA.

Characterizing spectra

Under some naturality conditions on $(\Omega, <)$:

Theorem

A sequence $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ is a conservativity spectrum of some theory T iff $\alpha_{i+1} \leq \ell(\alpha_i)$, for all $i \in \omega$.

Here $\ell(\beta) = 0$ if $\beta = 0$, and $\ell(\beta) = \gamma$ if $\beta = \delta + \omega^\gamma$, for some δ, γ.

Remarks

- The set of all spectra $\vec{\alpha}$ such that $\forall i \in \omega \, \alpha_i < \varepsilon_0$ is the domain of the *Ignatiev model*, a well-known universal Kripke model $\mathcal{I}$ for the closed fragment of Japaridze's provability logic GLP.
- *J. Joosten (2016)* has shown the injection of $\mathcal{I}$ into the set of spectra of subtheories of PA.

λ-conservativity spectra

Extended framework (jww with Pakhomov (2022))

1. Expand the language of PA by truth definitions $T_\alpha(x)$, $\alpha < \lambda$.
2. Conservatively extend EA by Tarski biconditionals: $\forall x\,(T_\alpha(\ulcorner\varphi(\dot{x})\urcorner) \leftrightarrow \varphi(x))$ for $\varphi \in \mathcal{L}(\{T_\beta : \beta < \alpha\})$.
3. Classes $\Pi_{1+\alpha}$ and reflection operators R_α, for all $\alpha < \lambda$.
4. Reflection algebras $(\mathfrak{G}_S; \wedge_S, 1_S, \{R_\alpha : \alpha < \lambda\})$.
5. $\Pi_{1+\alpha}$-ordinals $\mathrm{ord}_\alpha(T)$ and conservativity spectra $f : \lambda \to \Omega$.

Examples of spectra:

ACA : $(\varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_0}, \dots; \varepsilon_0, \varepsilon_0, \dots; 0, \dots)$

$\mathrm{ACA}^+ := \mathrm{ACA} + \forall X\, \exists Y\; Y = X^{(\omega)} \equiv \mathrm{PA}(T_0, T_1, \dots, T_\omega)$.

Spectrum: $f(\alpha) = \varphi_2(\varepsilon_0)$ if $\alpha < \omega^2$;
$f(\alpha) = \varepsilon_0$ if $\omega^2 \le \alpha < \omega^2 + \omega$.

Extended framework (jww with Pakhomov (2022))

1. Expand the language of PA by truth definitions $T_\alpha(x)$, $\alpha < \lambda$.
2. Conservatively extend EA by Tarski biconditionals: $\forall x\,(T_\alpha(\ulcorner\varphi(\dot{x})\urcorner) \leftrightarrow \varphi(x))$ for $\varphi \in \mathcal{L}(\{T_\beta : \beta < \alpha\})$.
3. Classes $\Pi_{1+\alpha}$ and reflection operators R_α, for all $\alpha < \lambda$.
4. Reflection algebras $(\mathfrak{G}_S; \wedge_S, 1_S, \{R_\alpha : \alpha < \lambda\})$.
5. $\Pi_{1+\alpha}$-ordinals $\mathrm{ord}_\alpha(T)$ and conservativity spectra $f : \lambda \to \Omega$.

Examples of spectra:

ACA : $\quad (\varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_0}, \dots; \varepsilon_0, \varepsilon_0, \dots; 0, \dots)$

$\mathrm{ACA}^+ := \mathrm{ACA} + \forall X\,\exists Y\; Y = X^{(\omega)} \equiv \mathrm{PA}(T_0, T_1, \dots, T_\omega)$.

Spectrum: $f(\alpha) = \varphi_2(\varepsilon_0)$ if $\alpha < \omega^2$;

$f(\alpha) = \varepsilon_0$ if $\omega^2 \leq \alpha < \omega^2 + \omega$.

Theorem

$f : \lambda \to \Omega$ is a λ-conservativity spectrum iff, for all α, β such that $\alpha + \omega^\beta < \lambda$,

1. $\ell(f(\alpha)) \geq f(\alpha + 1)$;
2. $\ell(f(\alpha)) \geq \varphi_\beta(f(\alpha + \omega^\beta))$ if $\beta > 0$.

Appeared in *Fernández–Joosten* (2014) as "ℓ-sequences".

Veblen functions

- $\varphi_0(\beta) := \omega^{1+\beta}$;
- $\varphi_{\alpha+1}(\beta) :=$ β-th fixed point of φ_α;
- $\varphi_\mu(\beta) :=$ β-th simultaneous f.p. of $\{\varphi_\alpha : \alpha < \mu\}$, if $\mu \in \mathrm{Lim}$.
- $\Gamma_0 :=$ the least ordinal > 0 closed under $\varphi_\alpha(\beta)$.

Theorem

$f : \lambda \to \Omega$ is a λ-conservativity spectrum iff, for all α, β such that $\alpha + \omega^\beta < \lambda$,

1. $\ell(f(\alpha)) \geq f(\alpha + 1)$;
2. $\ell(f(\alpha)) \geq \varphi_\beta(f(\alpha + \omega^\beta))$ if $\beta > 0$.

Appeared in *Fernández–Joosten* (2014) as "ℓ-sequences".

Veblen functions

- $\varphi_0(\beta) := \omega^{1+\beta}$;
- $\varphi_{\alpha+1}(\beta) := \beta$-th fixed point of φ_α;
- $\varphi_\mu(\beta) := \beta$-th simultaneous f.p. of $\{\varphi_\alpha : \alpha < \mu\}$, if $\mu \in \mathrm{Lim}$.
- $\Gamma_0 :=$ the least ordinal > 0 closed under $\varphi_\alpha(\beta)$.

Reflection calculus RC

Strictly positive formulas (= SLO terms)
$A ::= \top \mid p \mid (A \wedge A) \mid \Diamond_n A \quad$ for $n \in \omega$

Sequents $A \vdash B$ denote inequations $\forall \vec{x}\,(A(\vec{x}) \leq B(\vec{x}))$

RC rules:

1. $A \vdash A$; $\quad A \vdash \top$; $\quad$ if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
2. $A \wedge B \vdash A, B$; $\quad$ if $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$;
3. if $A \vdash B$ then $\Diamond_n A \vdash \Diamond_n B$; $\quad \Diamond_n \Diamond_n A \vdash \Diamond_n A$;
4. $\Diamond_n A \vdash \Diamond_m A$ for $n > m$;
5. $\Diamond_n A \wedge \Diamond_m B \vdash \Diamond_n (A \wedge \Diamond_m B)$ for $n > m$.

Main results on RC

Theorems (E. Dashkov, 2012)

1. $A \vdash_{RC} B$ iff $A \vdash B$ holds in $(\mathfrak{G}_{\mathrm{PA}}; \wedge_{\mathrm{PA}}, 1_{\mathrm{PA}}, \{R_n : n \in \omega\})$;
2. RC is polytime decidable;
3. RC enjoys the finite model property.

The first claim is based on Japaridze's (1986) arithmetical completeness theorem for provability logic GLP.

Main results on RC

Theorems (E. Dashkov, 2012)

1. $A \vdash_{RC} B$ iff $A \vdash B$ holds in $(\mathfrak{G}_{\mathrm{PA}}; \wedge_{\mathrm{PA}}, 1_{\mathrm{PA}}, \{R_n : n \in \omega\})$;
2. RC is polytime decidable;
3. RC enjoys the finite model property.

The first claim is based on Japaridze's (1986) arithmetical completeness theorem for provability logic GLP.

RC^0 as an ordinal notation system

Let RC^0 denote the variable-free fragment of RC.
Let W_n denote the set of all RC^0-formulas with $\Diamond_i$ for $i \geq n$.
For $A, B \in W_n$ define:

- $A \sim B$ if $A \vdash B$ and $B \vdash A$ in RC^0;
- $A <_n B$ if $B \vdash \Diamond_n A$.

Theorem

1. Every $A \in W_n$ is equivalent to a *word* (formula without $\wedge$);
2. $(W_n/{\sim}, <_n)$ is isomorphic to $(\varepsilon_0, <)$.

We consider operators associating with a theory T the theory generated by its consequences of logical complexity Π_{n+1}:

$$\Pi_{n+1}(T) := \{\pi \in \Pi_{n+1} : S \vdash \pi\}.$$

Notice that each Π_{n+1} is a closure operator.

RC^∇ algebra of S

$(\mathfrak{G}_S; \wedge_S, 1_S, \{R_n, \Pi_{n+1} : n \in \omega\})$

Open problem:

Characterize the logic/identities of this structure. Is it (polytime) decidable?

The system RC^{∇}

RC^{∇} is a strictly positive logic with modalities $\{\Diamond_n, \nabla_n : n \in \omega\}$ ($\Diamond_n$ for R_n, ∇_n for Π_{n+1}).

Axioms and rules:

1. RC for $\Diamond_n$;
2. RC for ∇_n;
3. $A \vdash \nabla_n A$; thus, each ∇_n satisfies $S4^+$;
4. $\Diamond_n A \vdash \nabla_n A$;
5. $\Diamond_m \nabla_n A \vdash \Diamond_m A$ if $m \leq n$;
6. $\nabla_n \Diamond_m A \vdash \Diamond_m A$ if $m \leq n$.

Isomorphism theorem

Let $\mathfrak{G}^0_{\mathsf{EA}}$ denote the subalgebra of $(\mathfrak{G}_{\mathsf{EA}}; \wedge_{\mathsf{EA}}, 1_{\mathsf{EA}}, \{R_n, \Pi_{n+1} : n \in \omega\})$ generated by 1_{EA}.

Theorem

The following structures are isomorphic:

1. $\mathfrak{G}^0_{\mathsf{EA}}$;
2. The free 0-generated RC^{∇}-algebra;
3. $\mathfrak{I} = (I, \wedge_{\mathfrak{I}}, \{\Diamond^{\mathfrak{I}}_n, \nabla^{\mathfrak{I}}_n : n \in \omega\})$.

Isomorphism theorem

Let $\mathfrak{G}^0_{\mathrm{EA}}$ denote the subalgebra of $(\mathfrak{G}_{\mathrm{EA}}; \wedge_{\mathrm{EA}}, 1_{\mathrm{EA}}, \{R_n, \Pi_{n+1} : n \in \omega\})$ generated by 1_{EA}.

Theorem

The following structures are isomorphic:

1. $\mathfrak{G}^0_{\mathrm{EA}}$;
2. The free 0-generated RC^{∇}-algebra;
3. $\mathfrak{I} = (I, \wedge_{\mathfrak{I}}, \{\Diamond^{\mathfrak{I}}_n, \nabla^{\mathfrak{I}}_n : n \in \omega\})$.

Ignatiev RC^∇-algebra

Named after K. Ignatiev who introduced a universal Kripke model for Japaridze's logic based on sequences of ordinals (1993).

- I is the set of all ω-sequences $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ such that $\alpha_i < \varepsilon_0$ and $\alpha_{i+1} \leq \ell(\alpha_i)$, for all $i \in \omega$.
- $\vec{\alpha} \leq_{\mathfrak{I}} \vec{\beta} \iff \forall i\, \alpha_i \geq \beta_i$.

Fact. The ordering $(I, \leq_{\mathfrak{I}})$ is a meet-semilattice.

Ignatiev RC^{∇}-algebra

Named after K. Ignatiev who introduced a universal Kripke model for Japaridze's logic based on sequences of ordinals (1993).

- I is the set of all ω-sequences $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ such that $\alpha_i < \varepsilon_0$ and $\alpha_{i+1} \leq \ell(\alpha_i)$, for all $i \in \omega$.
- $\vec{\alpha} \leq_{\mathfrak{I}} \vec{\beta} \iff \forall i\, \alpha_i \geq \beta_i$.

Fact. The ordering $(I, \leq_{\mathfrak{I}})$ is a meet-semilattice.

Ignatiev RC^{∇}-algebra

We define the functions $\nabla_n^{\mathfrak{I}}, \Diamond_n^{\mathfrak{I}} : I \to I$.
For each $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ let

- $\nabla_n^{\mathfrak{I}}(\vec{\alpha}) := (\alpha_0, \alpha_1, \dots, \alpha_n, 0, \dots)$;
- $\Diamond_n^{\mathfrak{I}}(\vec{\alpha}) := (\beta_0, \beta_1, \dots, \beta_n, 0, \dots)$, where $\beta_{n+1} := 0$ and $\beta_i := \alpha_i + \omega^{\beta_{i+1}}$, for all $i \leq n$.

Fact. The SLO $\mathfrak{I} = (I, \wedge_{\mathfrak{I}}, \{\Diamond_n^{\mathfrak{I}}, \nabla_n^{\mathfrak{I}} : n \in \omega\})$ is an RC^{∇}-algebra.

Ignatiev RC^∇-algebra

We define the functions $\nabla_n^{\mathfrak{I}}, \Diamond_n^{\mathfrak{I}} : I \to I$.
For each $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ let

- $\nabla_n^{\mathfrak{I}}(\vec{\alpha}) := (\alpha_0, \alpha_1, \dots, \alpha_n, 0, \dots)$;
- $\Diamond_n^{\mathfrak{I}}(\vec{\alpha}) := (\beta_0, \beta_1, \dots, \beta_n, 0, \dots)$, where $\beta_{n+1} := 0$ and $\beta_i := \alpha_i + \omega^{\beta_{i+1}}$, for all $i \leq n$.

Fact. The SLO $\mathfrak{I} = (I, \wedge_{\mathfrak{I}}, \{\Diamond_n^{\mathfrak{I}}, \nabla_n^{\mathfrak{I}} : n \in \omega\})$ is an RC^∇-algebra.

An extension T of EA is *bounded*, if T is contained in a finite subtheory of PA.

Theorem

1. Let T be bounded and $\vec{\alpha}$ be the conservativity spectrum of T. Then $\forall n < \omega\ \alpha_{n+1} \leq \ell(\alpha_n)$ and $\alpha_n < \varepsilon_0$, that is, $\vec{\alpha} \in \mathfrak{I}$.
2. Let $\vec{\alpha} \in \mathfrak{I}$, A be a variable-free RC^{∇}-formula corresponding to $\vec{\alpha}$ via the isomorphism, and $A_{\mathrm{EA}} \in \mathfrak{G}^0_{\mathrm{EA}}$ its arithmetical interpretation. Then $\vec{\alpha}$ is the conservativity spectrum of A_{EA}.
3. A_{EA} is the weakest theory with the given conservativity spectrum $\vec{\alpha}$.

Spectra and $\mathfrak{I}$

An extension T of EA is *bounded*, if T is contained in a finite subtheory of PA.

Theorem

1. Let T be bounded and $\vec{\alpha}$ be the conservativity spectrum of T. Then $\forall n < \omega\ \alpha_{n+1} \leq \ell(\alpha_n)$ and $\alpha_n < \varepsilon_0$, that is, $\vec{\alpha} \in \mathfrak{I}$.
2. Let $\vec{\alpha} \in \mathfrak{I}$, A be a variable-free RC^{∇}-formula corresponding to $\vec{\alpha}$ via the isomorphism, and $A_{\mathrm{EA}} \in \mathfrak{G}^0_{\mathrm{EA}}$ its arithmetical interpretation. Then $\vec{\alpha}$ is the conservativity spectrum of A_{EA}.
3. A_{EA} is the weakest theory with the given conservativity spectrum $\vec{\alpha}$.

Spectra and $\mathfrak{I}$

An extension T of EA is *bounded*, if T is contained in a finite subtheory of PA.

Theorem

1. Let T be bounded and $\vec{\alpha}$ be the conservativity spectrum of T. Then $\forall n < \omega\ \alpha_{n+1} \leq \ell(\alpha_n)$ and $\alpha_n < \varepsilon_0$, that is, $\vec{\alpha} \in \mathfrak{I}$.
2. Let $\vec{\alpha} \in \mathfrak{I}$, A be a variable-free RC^∇-formula corresponding to $\vec{\alpha}$ via the isomorphism, and $A_{\mathrm{EA}} \in \mathfrak{G}^0_{\mathrm{EA}}$ its arithmetical interpretation. Then $\vec{\alpha}$ is the conservativity spectrum of A_{EA}.
3. A_{EA} is the weakest theory with the given conservativity spectrum $\vec{\alpha}$.

Encore: expressibility of iterations

Let $A \in W_n$ and $\alpha = o_n(A)$ denote its ordinal notation in $(W_n, <_n)$.

Theorem

In $\mathfrak{G}_{\mathrm{EA}}$, $\nabla_n A =_{\mathrm{EA}} \Diamond_n^{\alpha}(\top)$.

Theorem

For each $n < \omega$ and $0 < \alpha < \varepsilon_0$ there is an RC-formula $A(p)$ s.t.

$$\forall S \in \mathfrak{G}_{\mathrm{EA}}\ \Diamond_n^{\alpha}(S) =_{\mathrm{EA}} \nabla_n A(S).$$

For example, $\nabla_0 \Diamond_1 \Diamond_0 \varphi = \Diamond_0^{\omega} \varphi$.

Encore: expressibility of iterations

Let $A \in W_n$ and $\alpha = o_n(A)$ denote its ordinal notation in $(W_n, <_n)$.

Theorem

In $\mathfrak{G}_{\mathsf{EA}}$, $\nabla_n A =_{\mathsf{EA}} \Diamond_n^{\alpha}(\top)$.

Theorem

For each $n < \omega$ and $0 < \alpha < \varepsilon_0$ there is an RC-formula $A(p)$ s.t.

$$\forall S \in \mathfrak{G}_{\mathsf{EA}}\ \Diamond_n^{\alpha}(S) =_{\mathsf{EA}} \nabla_n A(S).$$

For example, $\nabla_0 \Diamond_1 \Diamond_0 \varphi = \Diamond_0^{\omega} \varphi$.