

# FEATURES OF (UN)DECIDABLE LOGICS

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# Plan for the talk

- (Un)decidability: what and why?
- Propositional team logics and their decidability
- Exploring boundaries between the decidable and the undecidable
  - Solving problems and obtaining insights along the way
  - Using insights to solve one last problem and provide a new proof of an old result

# (Un)decidability: what and why?

## What?

A **decision problem** is a collection of inputs  $I$ , with a yes-or-no question for each  $i \in I$ .

A decision problem is **decidable** if there is an effective method that, given any  $i \in I$ , accurately answers the question. Otherwise, it is **undecidable**.

A logic  $\mathbf{L}$ , in a language  $\mathcal{L}$ , is decidable if there is an algorithm that, given any  $\varphi \in \mathcal{L}$ , determines whether  $\varphi \in \mathbf{L}$ . Otherwise, it is undecidable.

## Why?

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## What?

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**Why?** *Because it is a deep and profound conceptual distinction.*

# Propositional team logics and their decidability

Traditionally (in, e.g., CPC), formulas  $\varphi$  are evaluated at **single valuations**  
 $v : \mathbf{Prop} \rightarrow \{0, 1\}$ ,

$$v \models \varphi.$$

In team semantics, formulas  $\varphi$  are evaluated at **sets ('teams') of valuations**  
 $s \subseteq \{v \mid v : \mathbf{Prop} \rightarrow \{0, 1\}\}$ ,

$$s \models \varphi.$$

## Definition (some team-semantic clauses)

Let  $X := \{v \mid v : \mathbf{Prop} \rightarrow \{0, 1\}\}$ . For  $s \in \mathcal{P}(X)$ , we define

$s \models p$	iff	$\forall v \in s : v(p) = 1$ ,
$s \models \varphi \wedge \psi$	iff	$s \models \varphi$ and $s \models \psi$ ,
$s \models \varphi \vee \psi$	iff	$s \models \varphi$ or $s \models \psi$ ,
$s \models \sim \varphi$	iff	$s \not\models \varphi$ ,
$s \models \varphi \cup \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \models \varphi$ ; $s'' \models \psi$ ; and $s = s' \cup s''$ .

**Observation.** All propositional team logics are decidable: given  $\varphi$ , simply check whether  $s \models \varphi$  for all  $s \subseteq \{v \mid v : \mathbf{Prop}(\varphi) \rightarrow \{0, 1\}\}$ .

Yet, this explanation is hardly satisfactory.  
What is it that makes propositional team logics  
decidable, *and others not?*

# Team semantics as relational semantics

Recall our semantic clauses: For  $X := \{v \mid v : \mathbf{Prop} \rightarrow \{0, 1\}\}$  and  $s \in \mathcal{P}(X)$ , we had

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This induces a **powerset frame**  $\mathbb{F} = (\mathcal{P}(X), \cup)$ , where ‘ $\circ$ ’ is a binary modality referring to the ternary  $\cup$ -relation:  $s = s' \cup s''$ ; and a **model**  $\mathbb{M} = (\mathcal{P}(X), \cup, V)$  with a ‘principal valuation’, i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = \downarrow\{v \in X \mid v(p) = 1\}.$$

**In fact**, if we take all powerset frames  $(\mathcal{P}(X), \cup)$ , redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \quad \text{iff} \quad s \in V(p),$$

and only allow principal valuations  $V : \mathbf{Prop} \rightarrow \{\downarrow s \mid s \in \mathcal{P}(X)\}$ , we get **sound and complete relational semantics for team logics**.

*Proof.* A simple p-morphism argument.

# Powerset frames and Boolean frames

Summarizing, (i) team logics are decidable, and (ii) relational semantics for team logics are given by powerset frames  $(\mathcal{P}(X), \cup)$  with principal valuations  $V : \mathbf{Prop} \rightarrow \{\downarrow s \mid s \in \mathcal{P}(X)\}$ .

**Question:** *Sticking with the signature  $\{\wedge, \vee, \neg, \circ\}$ , what happens if we allow for arbitrary valuations  $V : \mathbf{Prop} \rightarrow \mathcal{PP}(X)$ ? Does the logic remain decidable?*

In fact, this question is intimately related with an open problem: Goranko and Vakarelov (1999) consider the logic of Boolean frames – instead of a powerset  $\mathcal{P}(X)$ , the carrier is a Boolean algebra  $B$  – and raises the problem of its decidability.<sup>1</sup>

## Theorem

The logic of powerset frames, in the signature  $\{\wedge, \vee, \neg, \circ\}$ , with arbitrary valuations is **undecidable**. And so is the hyperboolean modal logic of Goranko and Vakarelov (1999).

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<sup>1</sup>Goranko and Vakarelov (1999) call their logic ‘hyperboolean modal logic’ and include modalities for all the Boolean operations, not just the join.

# Proof method: tiling

- A (Wang) tile is a square with colors on each side.
- **The tiling problem:** given any finite set of tiles  $\mathcal{W}$ , determine whether each point in the quadrant  $\mathbb{N}^2$  can be assigned a tile from  $\mathcal{W}$  such that neighboring tiles share matching colors on connecting sides.
- The tiling problem was introduced by Wang (1963) and proven **undecidable** by Berger (1966).

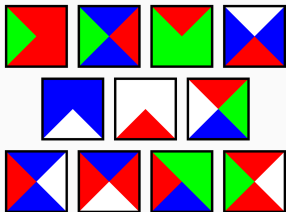


Figure 1: Wang tiles

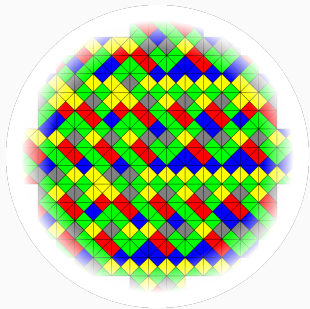


Figure 2: A tiling of the plane

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## Proof idea.

For each finite set of tiles  $\mathcal{W}$ , we construct a formula  $\phi_{\mathcal{W}}$  such that  $\mathcal{W}$  tiles the quadrant if and only if  $\phi_{\mathcal{W}}$  is satisfiable.  $\square$

Dividing the proof into two lemmas, corresponding to a direction each, we can prove both results in one go:

## Lemma

If  $\phi_{\mathcal{W}}$  is satisfiable (in a Boolean frame), then  $\mathcal{W}$  tiles  $\mathbb{N}^2$ .

## Lemma

If  $\mathcal{W}$  tiles  $\mathbb{N}^2$ , then  $\phi_{\mathcal{W}}$  is satisfiable (in  $(\mathcal{P}(\mathbb{N}), \cup)$ ).

Insight 1: **valuations** matter

# Semilattice frames, associativity and negation

**Question:** *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets  $(\mathcal{P}(X), \cup)$  to general (join-)semilattices  $(S, \sqcup)$ , we get a problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

## Theorem

For any class  $\mathcal{S}$  of semilattices containing  $(\mathcal{P}(\mathbb{N}), \cup)$ , its logic in the signature  $\{\wedge, \vee, \neg, \circ\}$ , is undecidable.

**Proof.** \*Formulas in handout (manuscript with proof available on request)\*



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**Question:** *What if we weaken even further than semilattices?*

**(Partial) answer:** As semilattices are partial orders ' $\leq$ ' with all binary suprema, we could consider the logic of all *partial orders simpliciter*. This is modal information logic, which is proven **decidable** in SBK (2023b).

**Question:** *What if we, instead, reduce our signature  $\{\wedge, \vee, \neg, \circ\}$ ?*

**Answer:** If we stick to semilattices but *omit negation*, so signature is  $\{\wedge, \vee, \circ\}$ , we obtain *Finean truthmaker semantics*, proven **decidable** in SBK (2023a).

Insight 2: **associativity** matters

Insight 3: **negation** matters

# (Un)decidability of relevant $\mathbf{S}$ : using our insights

**Problem of concern:** *Is relevant logic  $\mathbf{S}$  decidable?*

$\mathbf{S}$  is the logic of semilattice frames  $(S, \sqcup, \mathbf{0})$  with a bottom element  $\mathbf{0}$ , with arbitrary valuations, in the signature  $\{\wedge, \vee, \rightarrow\}$ . ' $\rightarrow$ ' is closely connected to ' $\circ$ ' (it is its residual).

**What we know about the problem:**

- Omitting disjunction, the logic  $\mathbf{S}_{\wedge, \rightarrow}$  is **decidable**.
- If we restrict to hereditary valuations, we obtain negation-free intuitionistic logic, which is **decidable**.
- $\mathbf{S}$  is closely connected to the relevant logic  $\mathbf{R}^+$ , which is **undecidable**.
  - Und. of  $\mathbf{R}^+$  was shown by Urquhart (1984), but  $\mathbf{S}$  eluded these techniques.
  - Eventually, this led experts, including Urquhart (2016), to conjecture that  $\mathbf{S}$  is **decidable**.

**What we notice about the problem:**

- *Valuations are arbitrary*, contra negation-free intuitionistic logic. [suggesting **undecidability**]
- $\mathbf{S}$  is *negation-free*! [suggesting **decidability**]
- Frames of  $\mathbf{S}$  are semilattices, *they are associative*! [suggesting **undecidability**]

**Theorem:** S is undecidable

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<sup>2</sup>See SBK (2024)

# Relevant S is undecidable: Proof idea

## Theorem: S is undecidable.

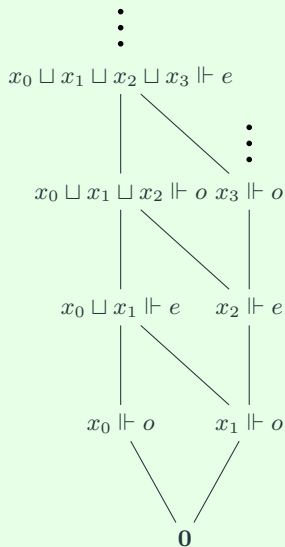
We cover the no-FMP proof instead, since it is considerably simpler than the undecidability proof, yet effectively illustrates some of the same key ideas.<sup>3</sup>

## Theorem: S lacks the FMP.

**Proof.** We show that the formula  $\psi_\infty$  from the handout only is refuted by infinite models.

<sup>3</sup> Additionally, it addresses an open problem (as recently raised in Weiss 2021)

## Refuting model





## Most recent research: What about **R**?

When Urquhart (1984) proved **R** (and **E** and **T**) undecidable, he concluded by remarking that “The undecidability results [...] omit one notable case. This is the logic consisting of all formulas valid in the semilattice semantics [...] The decision problem for this system is still open.”

# Most recent research: What about $\mathbf{R}$ ?

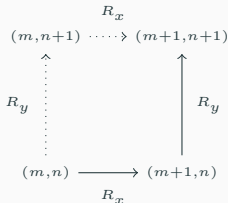
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**Question:** While  $\mathbf{S}$  escaped those techniques, can we extend the present proof to include  $\mathbf{R}$  (and  $\mathbf{E}$  and  $\mathbf{T}$ ) as well?

**Answer:** Yes! It extends to  $\mathbf{R}^+$ , hence  $\mathbf{R}$ , as well ( $\mathbf{E}$  and  $\mathbf{T}$  to be checked).

**Two comments on the proof:**

1. We use the  $n \times n$ , for all  $n \in \mathbb{N}$ , tiling problem instead.
2. On **associativity and tiling** (modulo commutativity):
  - Associativity for ternary relations:  $R(ab)cd \Rightarrow Ra(bc)d$ .<sup>4</sup>
  - Write  $aR_b c$  for  $Rabc$ . Then  $R(ab)cd$  means  $\exists e: aR_b eR_c d$ ; and  $R(ac)bd$  means  $\exists f: aR_c fR_b d$ .
  - So sp.  $(m, n)R_x(m+1, n)R_y(m+1, n+1)$ . From associativity, we get that there is a point  $(m, n+1)$  s.t.  $(m, n)R_y(m, n+1)R_x(m+1, n+1)$ . I.e.:



<sup>4</sup>Mod comm., it is  $R(ab)cd \Rightarrow R(ac)bd$ .  $R(ab)cd$  means  $\exists e: Rabe \wedge Recd$ ; and  $R(ac)bd$  means  $\exists f: Racf \wedge Rfbd$ .

# Conclusion

We obtained new (undecidability) results, including:

- Hyperboolean modal logic is **undecidable**.<sup>5</sup>
- Modal logic of semilattices is **undecidable**.<sup>6</sup>
- **S** is **undecidable** [cf. SBK 2024].<sup>7</sup>
- (and a new proof of Urquhart (1984)'s result that **R** is **undecidable**)

We compared them with known decidability results:

- Propositional team logics are **decidable**.
- Modal information logic is **decidable** [cf. SBK 2023b].
- Truthmaker logics are **decidable** [cf. SBK 2023a].

Core messages:

- **Valuations** matter.
- **Associativity** matters.
- **Negation** matters, but we only needed a tiny bit of meta-language 'it is not the case that'.<sup>8</sup>




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<sup>5</sup>Raised in Goranko and Vakarelov 1999

<sup>6</sup>Raised in Bergman 2018; Jipsen et al. 2021; SBK 2023b






<sup>7</sup>Raised by Urquhart 1972, 1984

<sup>8</sup>Or 'modal if-then' matters [ $\rightarrow$  and  $O(\varphi, \psi) := \neg\varphi \bullet \psi$ ] / (antitone, monotone) modal operator matters(?)

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Thank you!