Lattices of Transitive Tense Logics

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Outline

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The language $\mathscr{L}_\Box$ of modal logic defined as follows:

$$\mathscr{L}_\Box \ni \varphi ::= p \in \mathrm{Prop} \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi,$$

A normal modal logic is a set L of formulas which contains

- axioms of the classical propositional logic CPC,
- $\Box(p \to q) \to (\Box p \to \Box q)$

and closed under the following rules

(MP) from α and $\alpha \to \beta$ infer β

(Sub) from $\alpha(p_0, \cdots, p_n)$ infer $\alpha(\varphi_0, \cdots, \varphi_n)$

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Let K denote the minimal normal modal logic and NExt(K) the set of all normal modal logics

Kripke semantics of modal logics

A Kripke-frame is a pair $\mathfrak{F} = (X, R)$ where X is a non-empty set and R a binary relation on X

We write $\mathfrak{F} \models \varphi$ if the formula φ is valid in $\mathfrak{F}$

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For each class $\mathcal{K}$ of frames, we write $\mathrm{Log}(\mathcal{K})$ for the set of formulas

$$\{\varphi : \forall \mathfrak{F} \in \mathcal{K}(\mathfrak{F} \models \varphi)\}$$

For each modal logic L, we write $\mathrm{Fr}(L)$ for the class of frames

$$\{\mathfrak{F} : \forall \varphi \in L(\mathfrak{F} \models \varphi)\}$$

A modal logic L is Kripke-complete if $L = \mathrm{Log}(\mathrm{Fr}(L))$

Lattices of logics

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We define lattice-theoretical operations $\bigwedge$ and $\bigoplus$ as follows:

$$\bigwedge\{L_i : i \in I\} = \bigcap_{i\in I} L_i \text{ and } \bigoplus\{L_i : i \in I\} = \bigcap\{L : L \supseteq L_i \text{ for all} i \in I\}$$

The top element of $(\mathrm{NExt}(\mathsf{K}), \subseteq)$ is the set $\mathscr{L}_\Box$ of all modal formulas and the bottom element is K. We write NExt(K) for both $(\mathrm{NExt}(\mathsf{K}), \subseteq)$ and $(\mathrm{NExt}(\mathsf{K}), \bigwedge, \bigoplus, \mathscr{L}_\Box, \mathsf{K})$

For each normal modal logic L and formula $\varphi \in \mathscr{L}_\Box$, we write $L \oplus \varphi$ for the minimal normal modal logic containing $L \cup \{\varphi\}$. For example, we have

- $\mathscr{L}_\Box = \mathsf{K} \oplus p$
- $\mathsf{K4} = \mathrm{Log}(\{\mathfrak{F} : \mathfrak{F} \text{ is transitive}\}) = \mathsf{K} \oplus \Diamond\Diamond p \to \Diamond p$

Why lattices of logics?

The normal modal logic K is well-studied and we know a lot about it:

- K is complete w.r.t the class of all Kripke frames (Kripke-completeness)
- K is complete w.r.t the class of all finite Kripke frames (FMP)
- K has the Craig interpolation property
- K is finitely axiomatizable
- K is decidable
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We could also ask questions like:

(1) given a formula φ, is the normal modal logic $\mathsf{K} \oplus \varphi = \mathscr{L}_\Box$?

(2) given a formula φ, is there any $\Gamma \subseteq \mathscr{L}_\Box$ such that $L_1 = \mathsf{K} \oplus \varphi \neq \mathsf{K} \oplus \Gamma = L_2$ while $\mathsf{Fr}(L_1) = \mathsf{Fr}(L_2)$?

These problems are not just about K itself, but also about logics in NExt(K)

Decidability of consistency in NExt(K)

Actually (1) is decidable in the following sense

Theorem

The following set is decidable:

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The key observation is the following lemma ([Makinson, 1971]):

Lemma

NExt(K) *contains only two co-atoms*

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Is there any normal modal logics share the same class of frames with Log($\bullet$)? How many?

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Degree of Kripke-incompleteness!

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Let $\mathcal{L}$ be a lattice of logics and $\equiv_{\mathsf{Fr}}$ the equivalence relation on $\mathcal{L}$ such that

$$L_1 \equiv_{\mathsf{Fr}} L_2 \text{ iff } L_1 \text{ shares the same class of frames as } L_2, \text{ i.e., } \mathsf{Fr}(L_1) = \mathsf{Fr}(L_2).$$

For each $L \in \mathcal{L}$, let

$$[L]_{\equiv_{\mathsf{Fr}}} := \{L' \in \mathcal{L} : \mathsf{Fr}(L) = \mathsf{Fr}(L')\}$$

The *degree of Kripke-incompleteness* $\deg_{\mathcal{L}}(L)$ *of* L *in* $\mathcal{L}$ is defined to be the cardinality of $[L]_{\equiv_{\mathsf{Fr}}}$

L is said to be *strictly Kripke-complete in* $\mathcal{L}$ if $\deg_{\mathcal{L}}(L) = 1$

Blok's dichotomy theorem on Kripke-incompleteness

One of the most important results on Kripke-incompleteness in NExt(K) [Blok, 1978]:

- every modal logic $L \in \mathrm{NExt}(\mathrm{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$

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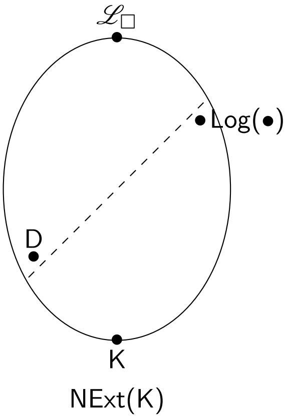
- union-splittings in NExt(K) are exactly the strictly Kripke-complete logics

Splittings

Let $\mathcal{L}$ be a lattice of logics and $L_1, L_2 \in \mathcal{L}$. Then $\langle L_1, L_2 \rangle$ is called a splitting pair in $\mathcal{L}$ if,

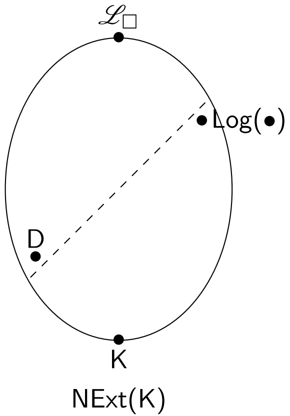
$$\text{for all } L \in \mathcal{L}, \text{ exactly one of } L \subseteq L_1 \text{ and } L \supseteq L_2 \text{ holds}$$

We say L_1 splits the lattice $\mathcal{L}$ and we call L_2 the splitting of $\mathcal{L}$ by L_1 and denote it by $\mathcal{L}/L_1$ (if $\mathcal{L} = \mathsf{NExt}(L_0)$, we also write L_0/L_1)



Union-splittings

L is called a *union-splitting* in $\mathcal{L}$ if $L = \bigoplus_{i \in I} L_i$ for some family $\{L_i : i \in I\}$ of splittings



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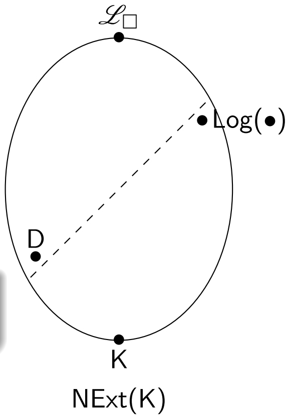
Theorem ([Takahashi, 2025])

The set $\{\varphi : \mathsf{K} \oplus \varphi$ *is a union splitting*$\}$ *is decidable*

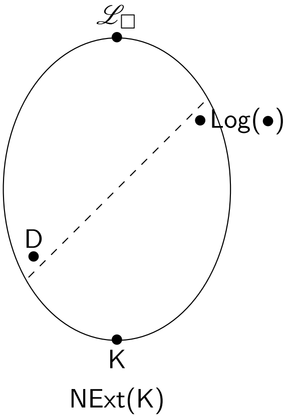


Iterated splittings

L is called a *iterated splitting* in $\mathcal{L}$ if $L = \mathcal{L}/L_1/L_2/\cdots/L_n$ for some $L_1, L_2, \cdots, L_n$ such that L_1 splits $\mathcal{L}$ and for all $1 \leq i < n$

$$L_{i+1} \text{ splits the lattice } \mathrm{NExt}(\mathcal{L}/L_1/\cdots/L_i)$$

(specially, if $\mathcal{L} = \mathrm{NExt}(L_0)$, L_0 is also an iterated splitting)



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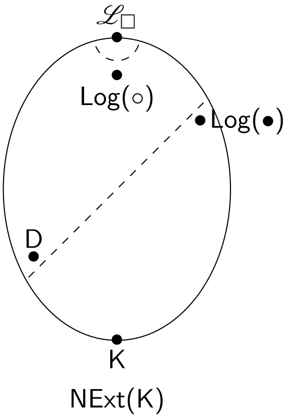
For example, $\mathscr{L}_\Box = \mathrm{K}/\mathrm{Log}(\bullet)/\mathrm{Log}(\circ)$ is an iterated splitting



Blok's theorem on Kripke-incompleteness

Blok's theorem on Kripke-incompleteness in NExt(K) [Blok, 1978]:

- every modal logic $L \in \mathrm{NExt}(\mathrm{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$
- union-splittings in NExt(K) are exactly the consistent strictly Kripke-complete logics
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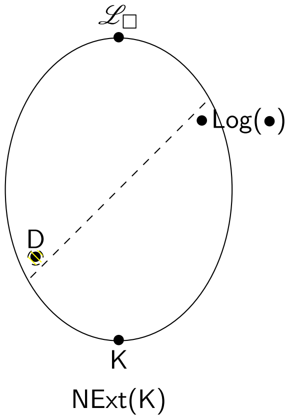

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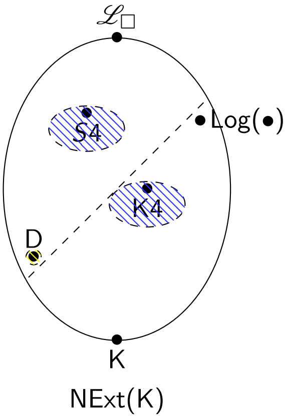

Decidability of strict Kripke-completeness

Now we have an answer to (2)

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Strict Kripke-completeness is decidable in NExt(K)*, that is, the following set is decidable:*

$$\{\varphi : \mathsf{K} \oplus \varphi \textit{ is strictly Kripke-complete}\}$$

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The observation is

Lemma

$$\{\varphi : \mathsf{K} \oplus \varphi \text{ is strictly Kripke-complete}\} = \{\varphi : \mathsf{K} \oplus \varphi = \mathscr{L}_\Box\} \cup \{\varphi : \mathsf{K} \oplus \varphi \text{ is a union-splitting}\}$$

More on the Degree of Incompleteness

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- Modal algebras MA: every normal modal logic is strictly MA-complete
- Neighborhood frames NF: [Chagrova, 1998], [Dziobiak, 1978] and [Litak, 2004] ...

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Degree of incompleteness in different lattices of logics, instead of NExt(K)

- A longstanding open problem: Does Blok's dichotomy theorem hold for K4, S4, or IPC?
- [Fornasiere and Moraschini, 2024]: Degrees of Kripke-incompleteness of implicative logics:
  - the trichotomy theorem: the degree is one of 1, $\aleph_0$ and $2^{\aleph_0}$

More on the Degree of Incompleteness

We could also do some combination:

- [Bezhanishvili et al., 2025]: Anti-dichotomy theorem of the degree of FMP for K4, S4, or IPC: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists L of degree of FMP κ

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Now, let's move to *tense logics*

Tense logics

The language $\mathscr{L}_t$ of tense logics is defined as follows:

$$\mathscr{L}_t ::= p \in \mathrm{Prop} \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi \mid \blacklozenge\varphi$$

A tense logic is a set of $\mathscr{L}_t$-formulas $L \supseteq \mathrm{CPC}$ closed under (MP) and (Sub), which satisfies

$$\text{(Adj) for all } \varphi, \psi \in \mathscr{L}_t,\ \blacklozenge\varphi \to \psi \in L \text{ if and only if } \varphi \to \Box\psi \in L$$

Let K_t be the minimal tense logic, $\mathrm{K4}_t = \mathrm{K}_t \oplus \Diamond p \to \Diamond\Diamond p$ and $\mathrm{S4}_t = \mathrm{K4}_t \oplus p \to \Diamond p$

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Fact

For all tense logic L, the following holds:

- $\Box\top \in L$ *and* $\Box\varphi \wedge \Box\psi \leftrightarrow \Box(\varphi \wedge \psi) \in L$
- $\blacklozenge\bot \leftrightarrow \bot \in L$ *and* $\blacklozenge\varphi \vee \blacklozenge\psi \leftrightarrow \blacklozenge(\varphi \vee \psi) \in L$
- $p \to \Box\blacklozenge p \in L$ *and* $p \to \blacksquare\Diamond p \in L$

Kripke frames and general frames for tense logics

A general frame is a tuple $\mathbb{F} = (X, R, A)$ where A is a subset of $\mathcal{P}(X)$ such that

- $\varnothing \in A$
- $U, V \in A$ implies $U \cap V \in A$ and $X \setminus U \in A$
- $U \in A$ implies $R[U] \in A$ and $R^{-1}[U] \in A$

A model is a tuple $\mathfrak{M} = (X, R, V)$ where $V : \mathrm{Prop} \to A$ is called a valuation in $\mathbb{F}$
For all point $x \in X$ and formula $\varphi \in \mathscr{L}_t$,

$$\mathfrak{M}, w \models \blacklozenge\varphi \text{ if and only if } \mathfrak{M}, u \models \varphi \text{ for some } u \in R^{-1}[w]$$

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K_t, $\mathrm{K4}_t$ and $\mathrm{S4}_t$ are the tense logics of the class of all Kripke frames, transitive Kripke frames and posets, respectively

Kripke semantics of tense logics

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Consider the following models

x and y satisfy the same $\mathscr{L}_\Box$ formulas, while $x \models \blacklozenge\Diamond p$ and $y \not\models \blacklozenge\Diamond p$

Lattices of tense logics

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There are countably many splittings in NExt(K) [Blok, 1978], while
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Degree of Kripke-incompleteness in lattices of tense logics?

Degree of Kripke-incompleteness in $\mathrm{NExt}(\mathrm{K4}_t)$ and $\mathrm{NExt}(\mathrm{K}_t)$

One result about the lattices $\mathrm{NExt}(\mathrm{K4}_t)$ and $\mathrm{NExt}(\mathrm{K}_t)$ from our previous work [C, 2025]

Theorem (Blok's Theorem for $\mathrm{NExt}(\mathrm{K}_t)$ and $\mathrm{NExt}(\mathrm{K4}_t)$)

Let $L \in \mathrm{NExt}(\mathrm{K}_t)$ (or $L \in \mathrm{NExt}(\mathrm{K4}_t)$). Then the following are equivalent:

- *L is a union-splitting*
- $\deg(L) = 1$
- $\deg(L) \neq 2^{\aleph_0}$

Degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$

We also studied the degree of Kripke-incompleteness of $\mathsf{NExt}(\mathsf{S4}_t)$ and showed the following:

Theorem (Blok's Theorem for $\mathsf{NExt}(\mathsf{S4}_t)$)

Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. *Then the following are equivalent:*

- L *is an iterated splitting in* $\mathsf{NExt}(\mathsf{S4}_t)$
- $\deg(L) = 1$
- $\deg(L) \neq 2^{\aleph_0}$

Degree of Kripke-incompleteness in $\mathrm{NExt}(\mathrm{S4}_t)$

We also studied the degree of Kripke-incompleteness of $\mathrm{NExt}(\mathrm{S4}_t)$ and showed the following:

Theorem (Blok's Theorem for $\mathrm{NExt}(\mathrm{S4}_t)$)

Let $L \in \mathrm{NExt}(\mathrm{S4}_t)$*. Then the following are equivalent:*

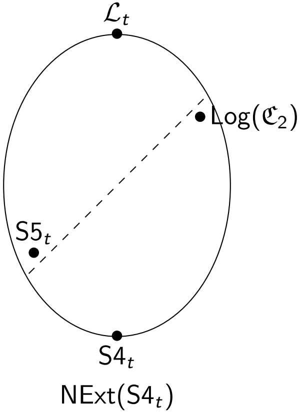
- L *is an iterated splitting in* $\mathrm{NExt}(\mathrm{S4}_t)$
- $\deg(L) = 1$
- $\deg(L) \neq 2^{\aleph_0}$

One more thing: there exists $L \in \mathrm{NExt}(\mathrm{S4}_t)$ which is not a union-splitting but with degree 1

Outline

Iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$

Recall from [Kracht, 1992] that $\langle \mathsf{Log}(\mathfrak{C}_2), \mathsf{S5}_t\rangle$ and $\langle \mathsf{Log}(\circ), \mathcal{L}_t\rangle$ are the only two splitting pairs in $\mathsf{NExt}(\mathsf{S4}_t)$, where $\mathfrak{C}_2 = (2, \leq)$ and $\mathsf{S5}_t = \mathsf{S4}_t \oplus (p \to \Box\Diamond p)$



Iterated splittings in NExt($S4_t$)

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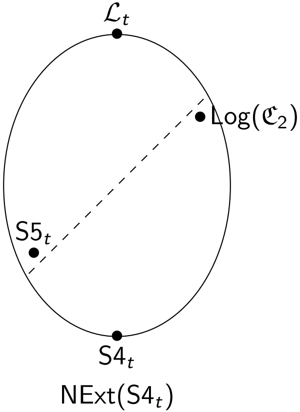
Lemma

Let $L \in \mathrm{NExt}(S4_t)$. *Then the following are equivalent:*

- L *is an iterated splitting in* NExt($S4_t$)
- $L \in \mathrm{NExt}(S5_t) \cup \{S4_t\}$



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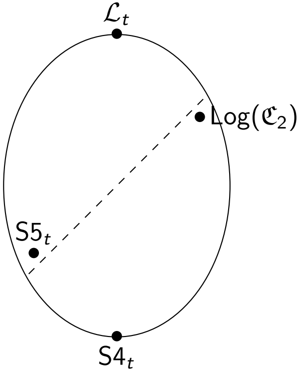

Lemma

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- $L \in \mathrm{NExt}(\mathrm{S5}_t) \cup \{\mathrm{S4}_t\}$

Theorem

Every iterated splitting in $\mathrm{NExt}(\mathrm{S4}_t)$ *is strictly Kripke-complete*

Non-iterated splittings in NExt($S4_t$)

Take any non-iterated splitting L in NExt($S4_t$).
It suffices now to prove that $\deg(L) = 2^{\aleph_0}$

To make the idea precise, let us focus on a concrete logic $L_0 = S4_t \oplus \Box\Diamond p \to \Diamond\Box p$

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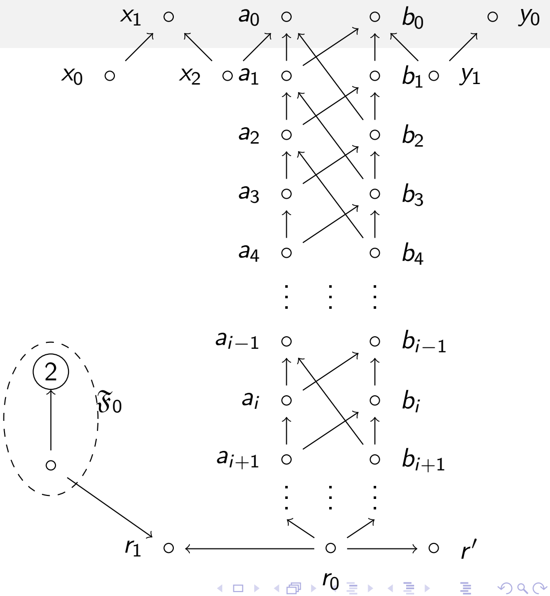
Consider the general frame $\mathbb{F} = (\mathfrak{F}, A)$ on the right hand side, where A is the set of all finite and co-finite subsets of X



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How to show that L_0 is not strictly complete?

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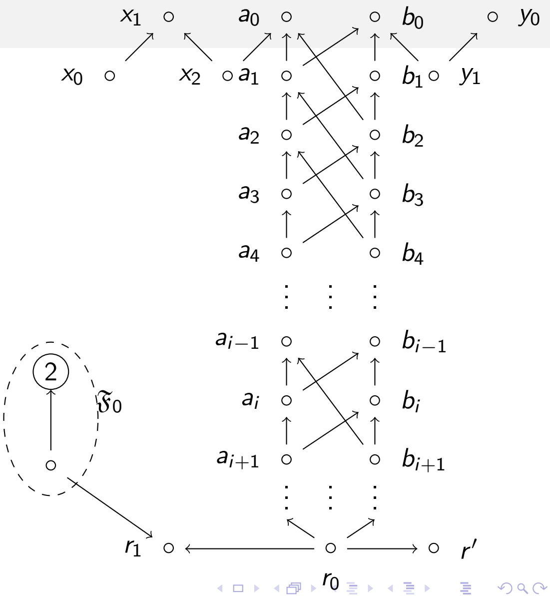
Let $L' = L_0 \cap \mathrm{Log}(\mathbb{F})$. Then $\Box\Diamond p \to \Diamond\Box p \notin L'$

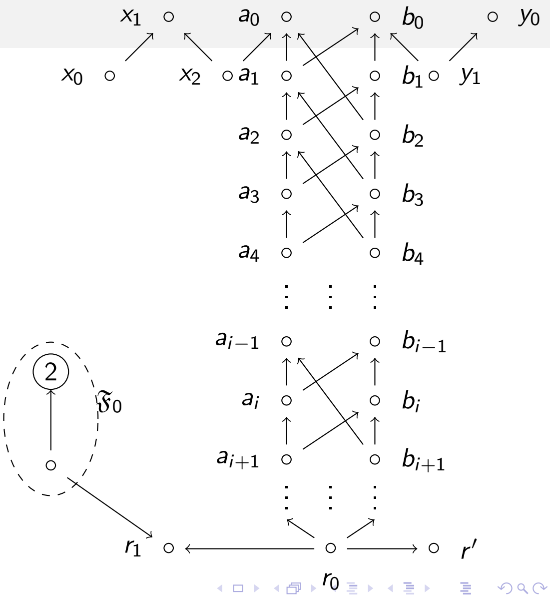


Non-iterated splittings in NExt($S4_t$)

The general $\mathbb{F}$ has several special properties:

Lemma

$\mathrm{Fin}_r(\mathrm{Log}(\mathbb{F})) = \mathrm{Fr}_r(\mathrm{Log}(\mathbb{F})) = \{\circ, \mathfrak{C}_2\}$.



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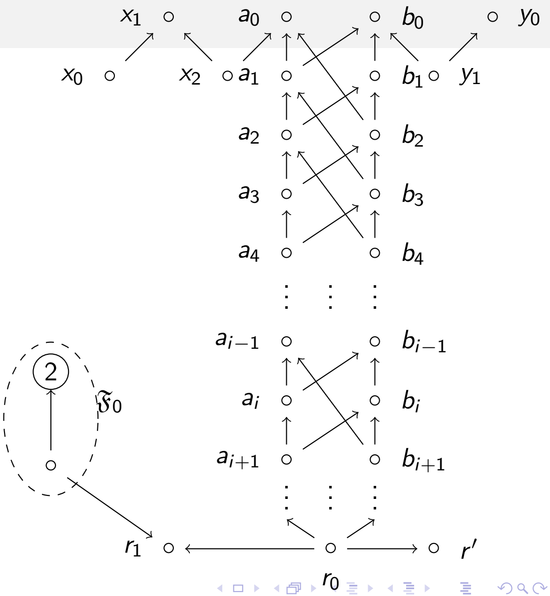
Lemma

$\mathrm{Fr}(L) = \mathrm{Fr}(L')$.

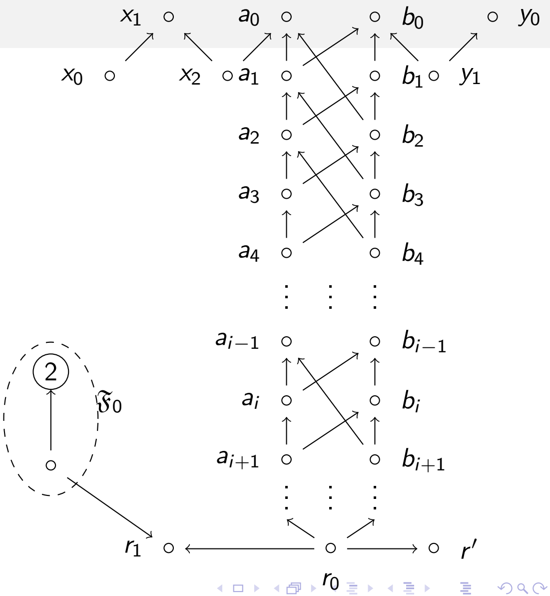
Theorem

$\deg(L_0) > 1$.



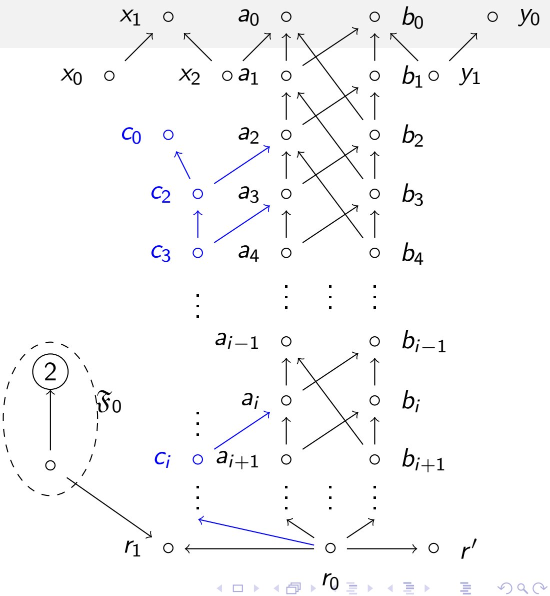
Non-iterated splittings in $\mathrm{NExt}(\mathrm{S4}_t)$

How to construct $2^{\aleph_0}$ logics in $[L_0]_{\mathrm{Fr}}$?



Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L_0]_{Fr}$?
For each $I \subseteq \omega$, construct $\mathbb{F}_I$ by adding points $\{c_i : i \in I\}$ and the corresponding arrows to $\mathbb{F}$:



Lemma

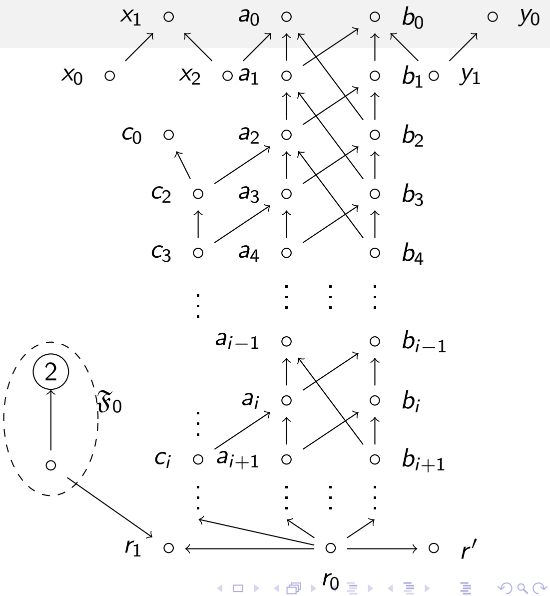
For all distinct $I, J \subseteq \mathbb{Z}^+$,

$$L_0 \cap \mathrm{Log}(\mathbb{F}_I) \neq L_0 \cap \mathrm{Log}(\mathbb{F}_J)$$

As a corollary, we see $\deg(L_0) = 2^{\aleph_0}$

Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L]_{Fr}$ for arbitrarily chosen non-iterated splitting L?



Non-iterated splittings in NExt($S4_t$)

How to construct $2^{\aleph_0}$ logics in $[L]_{\mathrm{Fr}}$ for arbitrarily chosen non-iterated splitting L?

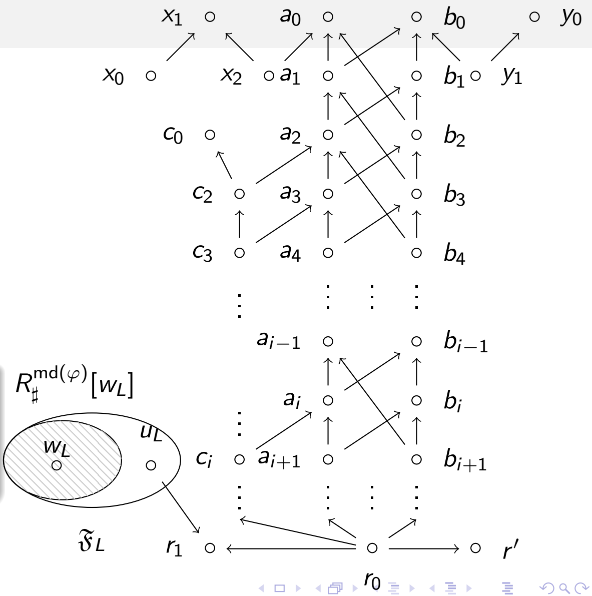
For each L, construct $\mathbb{F}^L_I$ by replacing $\mathfrak{F}_0$ with $\mathfrak{F}_L$, where $\mathfrak{F}_L \in \mathrm{Fin}_r(S4_t)$, $\varphi \in L \setminus S4_t$, $\mathfrak{F}_L, w_L \not\models \varphi$ and $u_L \notin R_\sharp^{\mathrm{md}(\varphi)}[w_L]$

Theorem

For all distinct $I, J \subseteq \mathbb{Z}^+$,

$$L \cap \mathrm{Log}(\mathbb{F}_I) \neq L \cap \mathrm{Log}(\mathbb{F}_J)$$

As a corollary, we see $\deg(L) = 2^{\aleph_0}$ for all non-iterated splittings



Final piece of our proof

Lemma

Let L be a proper extension of $\mathsf{S4}_t$ and $\varphi \in L \setminus \mathsf{S4}_t$. Then

- φ *is refuted by some non-symmetric* $\mathfrak{G} \in \mathsf{Fin}_r(\mathsf{S4}_t)$
- *for all* $n \in \omega$, *there exists* $\mathfrak{F} = (X, R) \in \mathsf{Fin}_r(\mathsf{S4}_t)$ *and* $x \in X$ *such that* $\mathfrak{F}, x \not\models \varphi$ *and* $(R \cup R^{-1})^n[x] \neq X$

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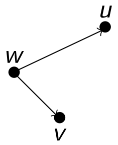

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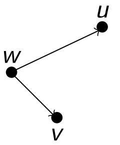

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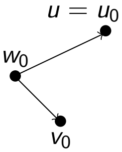

$\mathfrak{F}^1_{w,u}$

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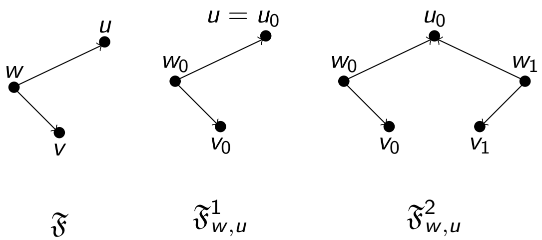

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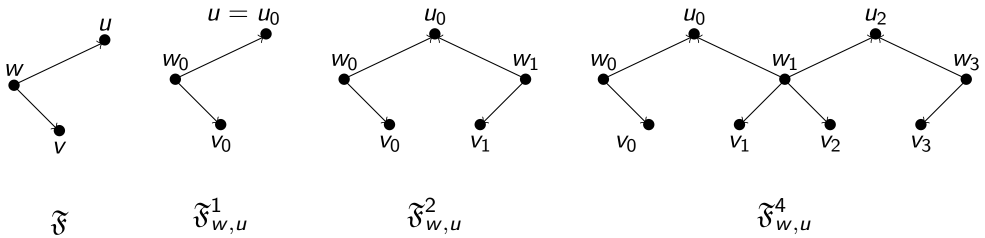

Conclusions

Blok's dichotomy theorem is generalized from NExt(K) (also NExt(K_t)) to NExt($S4_t$)

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Future works:

- Degree of Kripke-incompleteness for other lattices of logics, say, NExt($S4.3_t$) or Ext(IPC)
- Degree of incompleteness w.r.t other semantics, for example, topological semantics
- Back to the basic modal case :)

A short review of my research at the ILLC

Lattices of tense logics

I started with studying pretabularity in $\mathsf{NExt}(\mathsf{S4}_t)$

Theorem

For all $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{NExt}(\mathsf{S4}_t)$ such that there are exactly κ Kripke-complete pretabular extension of L

Then I turn to investigate the degree of incompleteness

Now I'm working with T. Takahashi on undecidability of logical properties:

Theorem

Consistency is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$

A short review of my research at the ILLC

Amalgamation and interpolation

I'm working on the amalgamation property in $\mathrm{NExt}(\mathrm{S5} \times \mathrm{S5})$ with N. Bezhanishvili

- We obtain a full characterization of logics in $\mathrm{NExt}(\mathrm{S5} \times \mathrm{S5})$ which have the amalgamation property
- There exists only 18 logics in $\mathrm{NExt}(\mathrm{S5} \times \mathrm{S5})$ having the amalgamation property

Amalgamation property in the lattice Ext(biGD) of bi-intuitionistic logics (with R. N. Almeida)

- We obtain a full characterization of logics in Ext(biGD) which have the amalgamation property

Thanks!

Bezhanishvili, G., Bezhanishvili, N., and Moraschini, T. (2025).
Degrees of the finite model property: The antidichotomy theorem.
Journal of Mathematical Logic.
DOI: 10.1142/S0219061325500060.

Blok, W. J. (1978).
On the degree of incompleteness in modal logic and the covering relation in the lattice of modal logics.
Technical Report 78-07.
University of Amsterdam.

Chagrova, L. (1998).
On the degree of neighborhood incompleteness of normal modal logics.
In *Advances in Modal Logic*, volume 1 of *CSLI Lecture Notes*, pages 63–72. CSLI Publications.

Chen, Q. and Ma, M. (2024).
Tabularity and post-completeness in tense logic.

The Review of Symbolic Logic, 17(2):475–492.

Chernev, A. (2022).
Degrees of fmp in extensions of bi-intuitionistic logic.
Master's thesis, University of Amsterdam, Amsterdam.

Dziobiak, W. (1978).
A note on incompleteness of modal logics with respect to neighbourhood semantics.
Bulletin of the Section of Logic, 7(4):185–189.

Fine, K. (1974).
An incomplete logic containing s4.
Theoria, 40(1):23–29.

Kracht, M. (1992).
Even more about the lattice of tense logics.
Archive for Mathematical Logic, 31(4):243–257.

Litak, T. (2004).

Modal incompleteness revisited.
Studia Logica, 76(3):329–342.

Makinson, D. (1971).
Some embedding theorems for modal logic.
Notre Dame Journal of Formal Logic, 12(2):252–254.

Thomason, S. K. (1972).
Semantic analysis of tense logics.
Journal of Symbolic Logic, 37(1):150–158.

Van Benthem, J. F. A. K. (1978).
Two simple incomplete modal logics.
Theoria, 44(1):25–37.