Connecting proof theory and semantics for non-normal modal logics

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Based on joint works with Charles Grellois, Nicola Olivetti, Björn Lellmann, Andrea Mazzullo, Sara Negri, Ana Ozaki, Elaine Pimentel, and Gian Luca Pozzato.

Outline

- Non-normal modal logics
- Bi-neighbourhood semantics
- Hypersequent calculus
- Applications

Non-normal modal logics

Lack some modal axioms or rules validated by the normal modal logic **K**.

Why non-normal modal logics

Normal modal logics are incompatible with possible interpretation of $\Box$:
Epistemic, deontic, agency, high probability, ...

Problems with monotonicity

$$RM \ \frac{A \to B}{\Box A \to \Box B}$$

- Epistemic logic and logical omniscience: "If someone knows Peano's axioms, then she knows that Fermat's conjecture is true."
- Deontic explosion: If a normative code contains a self-inconsistent obligation, then everything is obligatory.
- Deontic paradoxes: Gentle murder p., Ross p., good Samaritan p., p. of free choice permissions, ... (cf. McNamara 2006):

Norm:	$\Box\neg(Smith.Murders.John)$
Norm:	$Smith.Murders.John \to \Box(Smith.Murders.John.Gently)$
Fact:	$Smith.Murders.John$
Valid statement:	$Smith.Murders.John.Gently \to Smith.Murders.John$
By RM:	$\Box(Smith.Murders.John.Gently) \to \Box(Smith.Murders.John)$
Consequence:	$\Box(Smith.Murders.John)$

$$C \quad \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$$

- Deontic logic and conflicting obligations: Contradicting obligations do not imply the obligation to realise a contradiction.
- Agency logic and incompatible actions: Possibility to do A and possibility to do B does not imply possibility to do $A \wedge B$.
- Majority logic: In most cases A and in most cases B does not imply in most cases $A \wedge B$.

Problems with necessitation

$$RN \ \frac{A}{\Box A}$$

- Epistemic logic and omniscience: The agent knows all valid statements.
- Deontic logic: All tautologies are obligatory.

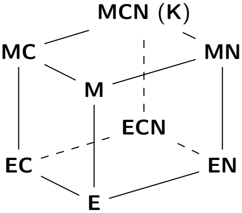
$$A ::= p \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A.$$

Basic system

$$\textbf{CPL} \quad + \quad RE\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

Extensions by adding any combination of:

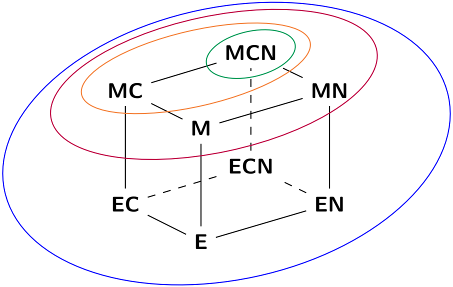
$M \quad \Box(A \wedge B) \rightarrow \Box A$	or $RM\ \dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$
$C \quad \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$	
$N \quad \Box\top$	or $RN\ \dfrac{A}{\Box A}$



- 8 non-equivalent systems.
- $M/C/N$ derivable only if they explicitly belong to the axiomatisation.
- Top system coincides with **K**.

A modal logic is

- congruential if it contains *RE*;
- monotonic if it contains *RE* and *M*;
- regular if it contains *RE*, *M*, and *C*;
- normal if it contains *RE*, *M*, *C*, and *N*.



Non-normal modal logics: Extending the classical cube

The classical cube can be extended with further principles

Axioms

$T \quad \Box A \rightarrow A \qquad D \quad \neg(\Box A \wedge \Box \neg A) \qquad P \quad \neg \Box \bot$

Rules

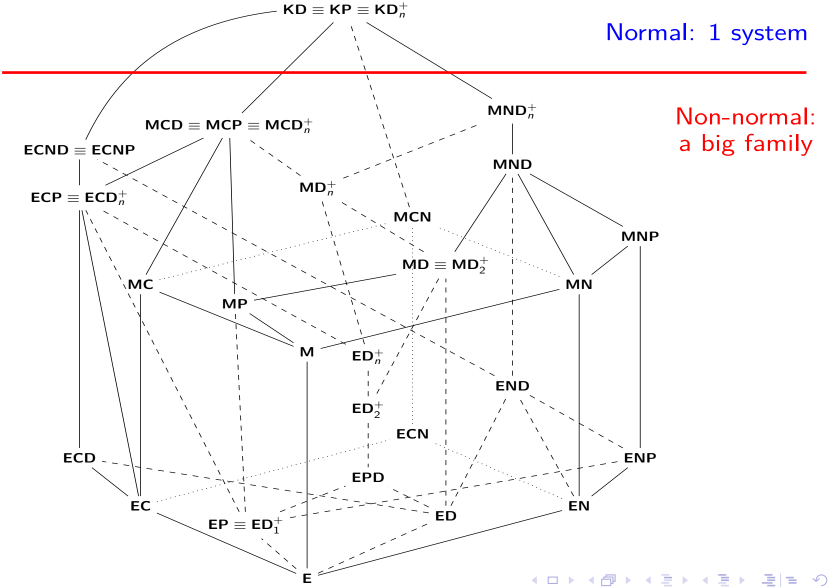
$$RD_2^+ \ \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Box B)} \qquad RD_3^+ \ \frac{\neg(A \wedge B \wedge C)}{\neg(\Box A \wedge \Box B \wedge \Box C)}$$

$$RD_4^+ \ \frac{\neg(A \wedge B \wedge C \wedge D)}{\neg(\Box A \wedge \Box B \wedge \Box C \wedge \Box D)} \qquad \ldots$$

- T: Factivity of knowledge, success of agent actions
- D, P, RD_n^+: No contradicting/self-inconsistent/incompatible obligations.

Remark: Non-iterative axioms/rules.
With iteration of modalities (e.g. axioms 4, 5, B) it gets more complicated.

"Deontic" non-normal modal logics with D, P, RD_2^+, RD_3^+, ...



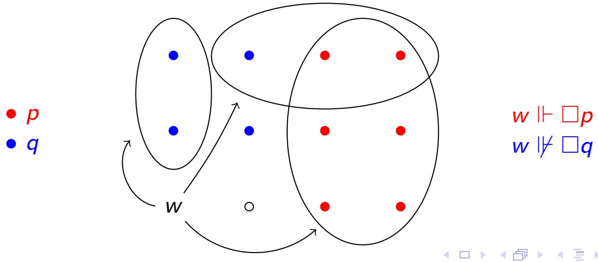
Standard neighbourhood models

$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where

- $\mathcal{W}$ non-empty set of worlds.
- $\mathcal{V}$ valuation function $Atm \longrightarrow \mathcal{P}(W)$.
- $\mathcal{N}$ neighbourhood function $\mathcal{W} \longrightarrow \mathcal{PP}(W)$.

Intuition: $\mathcal{N}$ assigns to every world the formulas which are necessary/known/obligatory/... in it:

$$w \Vdash \Box A \quad \text{iff} \quad [\![A]\!] \in \mathcal{N}(w)$$



Standard neighbourhood semantics (Scott 1970, Montague 1970)

Standard neighbourhood models

$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where

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- $\mathcal{N}$ neighbourhood function $\mathcal{W} \longrightarrow \mathcal{PP}(W)$.

$$w \Vdash \Box A \quad \text{iff} \quad [\![A]\!] \in \mathcal{N}(w)$$

Model conditions for extensions

(M)	If $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}(w)$.
(C)	If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$.
(N)	$\mathcal{W} \in \mathcal{N}(w)$.
(T)	If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$.
(P)	$\emptyset \notin \mathcal{N}(w)$.
(D)	If $\alpha \in \mathcal{N}(w)$, then $\mathcal{W} \setminus \alpha \notin \mathcal{N}(w)$.
(RD_n^+)	If $\alpha_1, ..., \alpha_n \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.

Characterisation (Chellas 1980)

$\mathcal{C}^{st}_{\mathbf{E}^*} \models A \quad$ iff $\quad \mathbf{E}^* \vdash A$.

Standard neighbourhood models

$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where

- $\mathcal{W}$ non-empty set of worlds.
- $\mathcal{V}$ valuation function $Atm \longrightarrow \mathcal{P}(W)$.
- $\mathcal{N}$ neighbourhood function $\mathcal{W} \longrightarrow \mathcal{PP}(W)$.

$$w \Vdash \Box A \quad \text{iff} \quad [\![A]\!] \in \mathcal{N}(w)$$

Behaves badly with nesting of modalities

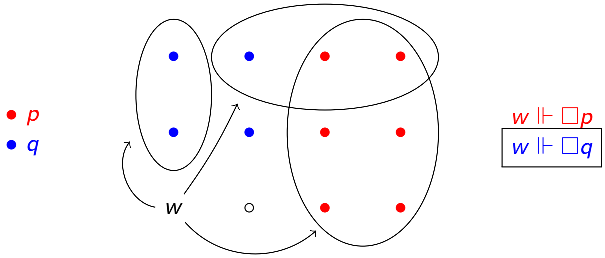
(4) If $\alpha \in \mathcal{N}(w)$, then $\{v \mid \alpha \in \mathcal{N}(v)\} \in \mathcal{N}(w)$.
(5) If $\alpha \notin \mathcal{N}(w)$, then $\{v \mid \alpha \notin \mathcal{N}(v)\} \in \mathcal{N}(w)$.
(B) If $w \in \alpha$, then $\{v \mid \mathcal{W} \setminus \alpha \notin \mathcal{N}(v)\} \in \mathcal{N}(w)$.

∃∀-neighbourhood models for monotonic systems

$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where

- $\mathcal{W}$ non-empty set of worlds.
- $\mathcal{V}$ valuation function $Atm \longrightarrow \mathcal{P}(W)$.
- $\mathcal{N}$ neighbourhood function $\mathcal{W} \longrightarrow \mathcal{PP}(W)$.

$$w \Vdash \Box A \quad \text{iff} \quad \text{there is } \alpha \in \mathcal{N}(w) \text{ s.t. } \alpha \subseteq [\![A]\!].$$



Satisfiability problem

Given a formula A and a logic $\mathbf{E}^*$, establish whether A is satisfiable in a neighbourhood model for $\mathbf{E}^*$.

Complexity (Vardi 1989)

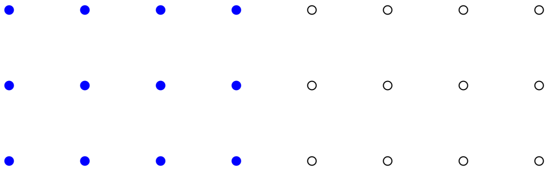
The satisfiability problem for $\mathbf{E}^*$ is

- NP-complete for the logics without without axiom C;
- in PSPACE with axiom C (explicit hardness for $\mathbf{MC}^*$).

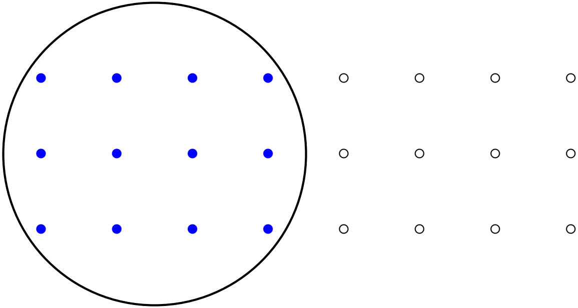
Bi-neighbourhood semantics
for reasoning with partial information

Bi-neighbourhood semantics

• p ○ $\neg p$



Bi-neighbourhood semantics



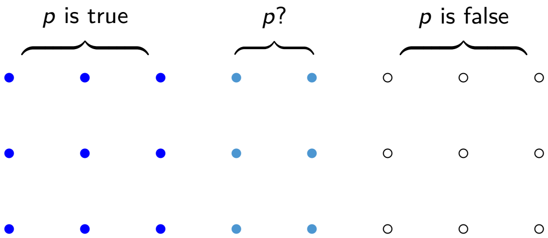
$$\in \mathcal{N}(w)$$

$$\Downarrow$$

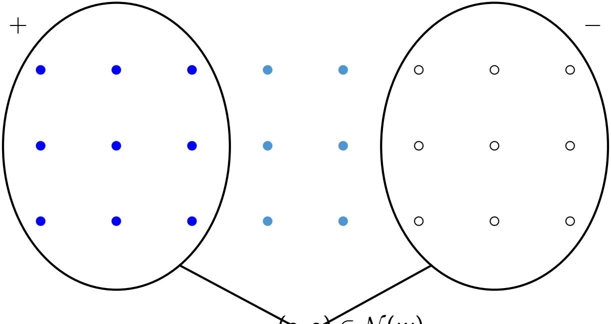
$$w \Vdash \Box p$$

Neighbourhood semantics requires exact determination of truth sets.

Bi-neighbourhood semantics



Bi-neighbourhood semantics

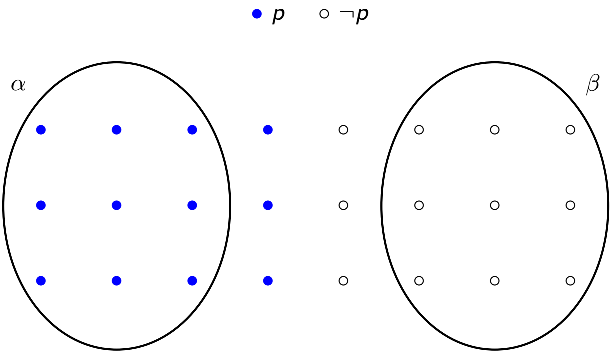


$(\bullet, \bullet) \in \mathcal{N}(w)$

$\Downarrow$

$w \Vdash \Box p$

Bi-neighbourhood semantics



$w \Vdash \Box p$ iff there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![p]\!]$ and $\beta \subseteq [\![\neg p]\!]$.

Bi-neighbourhood semantics

$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W} \neq \emptyset$; $\mathcal{V} : Atm \longrightarrow \mathcal{P}(W)$; and

- $\mathcal{N}$ bi-neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(W) \times \mathcal{P}(W))$.

$w \Vdash \Box A$ iff there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![A]\!]$ and $\beta \subseteq [\![\neg A]\!]$.

Conditions for extensions

(M)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\beta = \emptyset$.
(N)	There is $\alpha \subseteq \mathcal{W}$ such that for all $w \in \mathcal{W}$, $(\alpha, \emptyset) \in \mathcal{N}(w)$.
(C)	If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}(w)$.
(T)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $w \in \alpha$.
(P)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\alpha \neq \emptyset$.
(D)	If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $\alpha \cap \gamma \neq \emptyset$ or $\beta \cap \delta \neq \emptyset$.
(RD_n^+)	If $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.

Characterisation

$\mathcal{C}^{bi}_{\mathbf{E}^*} \models A$ iff $\mathbf{E}^* \vdash A$.

Bi-neighbourhood semantics

$$w \Vdash \Box A \quad \text{iff} \quad \text{there is } (\alpha, \beta) \in \mathcal{N}(w) \text{ s.t. } \alpha \subseteq [\![A]\!] \text{ and } \beta \subseteq [\![\neg A]\!].$$

Bi-neighbourhood models can be seen as

- A semantics for NNMLs.
- An underspecification of neighbourhood models:
  - (α, β) as lower and upper bounds of standard neighbourhoods:
  - Equivalent standard models definable with

$$\mathcal{N}_{st}(w) = \{\gamma \mid \text{there is } (\alpha, \beta) \in \mathcal{N}_{bi}(w) \text{ s.t. } \alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta\}.$$

- A reduction of congruential modality to a dyadic monotonic modality ($\exists\forall$-semantics):

$$\mathbf{M_2} := \mathbf{CPL} + \; RM_{\heartsuit} \; \frac{A \to C \qquad B \to D}{\heartsuit(A/B) \to \heartsuit(C/D)} \; .$$

$$(\Box A)^\circ = \heartsuit(A^\circ / \neg A^\circ).$$

$$\mathbf{E} \vdash A \quad \text{iff} \quad \mathbf{M_2} \vdash A^\circ.$$

Proof theory

- Study of logics from a purely syntactic point of view
- Establish properties of the logics by looking at the form of the proofs
- Implementation of automated theorem prover

Sequent calculus G3cp for classical logic

- Sequent: $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets (lists/sets) of formulas.
- $A_1, \dots, A_n \Rightarrow B$ "B is derivable from the assumptions $A_1, \dots, A_n$".
- Formula interpretation: $\iota(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$.

Rules

$$\text{init}\ \dfrac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \text{L}\bot\ \dfrac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \text{L}\neg\ \dfrac{\Gamma, \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \text{L}\wedge\ \dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\text{L}\vee\ \dfrac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \text{L}\to\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} \qquad \text{R}\neg\ \dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$\text{R}\vee\ \dfrac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \text{R}\wedge\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \qquad \text{R}\to\ \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

Derivation: Tree with initial sequents as leaves

$$\dfrac{\dfrac{\dfrac{\dfrac{q \to r, p \Rightarrow r, p \qquad \dfrac{q, p \Rightarrow r, q \qquad q, r, p \Rightarrow r}{q, q \to r, p \Rightarrow r}\,\text{L}\to}{p \to q, q \to r, p \Rightarrow r}\,\text{L}\to}{(p \to q) \wedge (q \to r), p \Rightarrow r}\,\text{L}\wedge}{(p \to q) \wedge (q \to r) \Rightarrow p \to r}\,\text{R}\to}{\Rightarrow ((p \to q) \wedge (q \to r)) \to (p \to r)}\,\text{R}\to$$

- **Sequent**: $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets (lists/sets) of formulas.
- $A_1, \ldots, A_n \Rightarrow B$ "B is derivable from the assumptions $A_1, \ldots, A_n$".
- **Formula interpretation**: $\iota(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Rules

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \text{L}\bot\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \text{L}\neg\ \frac{\Gamma, \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \text{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \text{L}\rightarrow\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \text{R}\neg\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$\text{R}\vee\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \qquad \text{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

Completeness

$\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathbf{CPL}} \bigwedge \Gamma \rightarrow \bigvee \Delta$

- Failed branches give countermodels

$$\dfrac{\dfrac{p \Rightarrow p \qquad p \Rightarrow q}{p \Rightarrow p \wedge q}\,\mathrm{R}\wedge \qquad \dfrac{q \Rightarrow p \qquad q \Rightarrow q}{q \Rightarrow p \wedge q}\,\mathrm{R}\wedge}{\dfrac{p \vee q \Rightarrow p \wedge q}{\Rightarrow p \vee q \to p \wedge q}\,\mathrm{R}\to}\,\mathrm{L}\vee$$

Countermodels:

(i) $p \mapsto 1,\ q \mapsto 0.$ (ii) $p \mapsto 0,\ q \mapsto 1.$

- (Un)derivability: A failed branch for $\Rightarrow A$ gives a countermodel for A.
- Satisfiability: A failed branch for $A \Rightarrow$ gives a model for A.

- Failed branches give countermodels

A key property: all rules are invertible

If the conclusion is derivable, then the premisses are derivable.

In concrete:

- Order of rule applications does not matter:

$$\dfrac{\mathrm{R}\wedge\ \dfrac{p \Rightarrow p \qquad p \Rightarrow q}{p \Rightarrow p \wedge q} \qquad \dfrac{q \Rightarrow p \qquad q \Rightarrow q}{q \Rightarrow p \wedge q}\ \mathrm{R}\wedge}{\dfrac{p \vee q \Rightarrow p \wedge q}{\Rightarrow p \vee q \rightarrow p \wedge q}\ \mathrm{R}\rightarrow}\ \mathrm{L}\vee$$

same as

$$\dfrac{\mathrm{L}\vee\ \dfrac{p \Rightarrow p \qquad q \Rightarrow p}{p \vee q \Rightarrow p} \qquad \dfrac{p \Rightarrow q \qquad q \Rightarrow q}{p \vee q \Rightarrow q}\ \mathrm{L}\vee}{\dfrac{p \vee q \Rightarrow p \wedge q}{\Rightarrow p \vee q \rightarrow p \wedge q}\ \mathrm{R}\rightarrow}\ \mathrm{R}\wedge$$

$\Rightarrow$ One proof search is enough to establish derivability/satisfiability.

- Failed branches give countermodels

A key property: all rules are invertible

If the conclusion is derivable, then the premisses are derivable.

- Order of rule applications does not matter:

$\Rightarrow$ One proof search is enough to establish derivability.

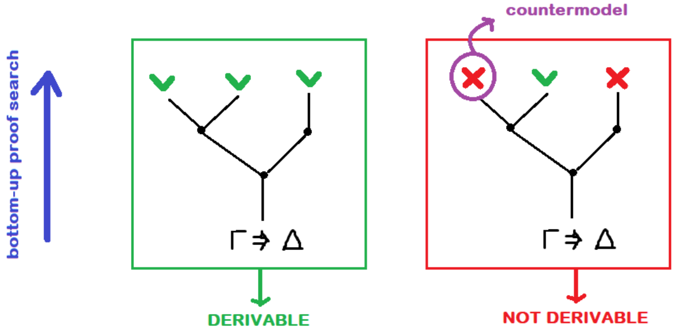
Complexity of derivability/satisfiability problem via proof search

- Branches have polynomial lenght w.r.t. the size of the initial sequent (key property: rules are analytic: the premisses have lower complexity than the conclusion).
- One failed branch is sufficient for satisfiability.

$\Rightarrow$ NP/coNP decision procedure for satisfiability/derivability.

Desiderata on proof system for NNMLs

- Terminating proof search procedure of optimal complexity.
- Countermodel generation: Obtain directly a countermodel from a single failed proof.



- Modular: Fixed set of basic rules & Extensions obtained by adding suitable rules.

G3cp extended with modal rules (Lavendhomme & Lucas 2000):

$$\mathsf{E}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \mathsf{M}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \mathsf{N}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta}$$

$$\mathsf{C}\ \frac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1 \quad ... \quad B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta} \qquad \mathsf{MC}\ \frac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta}$$

G3.E := E	**G3.M** := M
G3.EC := C	**G3.MC** := MC
G3.EN := E + N	**G3.MN** := M + N
G3.ECN := C + N	**G3.MCN** := MC + N.

- Structural rules admissible and syntactic cut elimination.
- Analyticity and termination of proof search.

G3cp extended with modal rules (Lavendhomme & Lucas 2000):

$$\text{E}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \text{M}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \text{N}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta}$$

$$\text{C}\ \frac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1 \quad ... \quad B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta} \qquad \text{MC}\ \frac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta}$$

But:

- Not modular: The calculi of stronger systems modify the rules for the weaker systems.

G3cp extended with modal rules (Lavendhomme & Lucas 2000):

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$$\mathsf{C}\ \frac{A_1, \dots, A_n \Rightarrow B \qquad B \Rightarrow A_1 \quad \dots \quad B \Rightarrow A_n}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \qquad \mathsf{MC}\ \frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}$$

But:

- Not modular.
- Modal rules are not invertible:
  - A single failed proof doesn't imply non-derivability.

non derivable

$$\dfrac{\dfrac{\dfrac{p, q \Rightarrow r}{p \wedge q \Rightarrow r}\ \mathsf{L}\wedge}{\Box(p \wedge q) \Rightarrow \Box r, \Box p}\ \mathsf{M}}{\Box(p \wedge q) \Rightarrow \Box r \vee \Box p}\ \mathsf{R}\vee \qquad \text{vs.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p, q \Rightarrow p}\ \mathsf{init}}{p \wedge q \Rightarrow q}\ \mathsf{L}\wedge}{\Box(p \wedge q) \Rightarrow \Box r, \Box p}\ \mathsf{M}}{\Box(p \wedge q) \Rightarrow \Box r \vee \Box p}\ \mathsf{R}\vee$$

Sequent calculi for non-normal modal logics

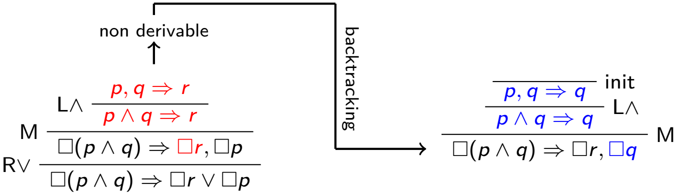
G3cp extended with modal rules (Lavendhomme & Lucas 2000):

$$\text{E}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \text{M}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \text{N}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta}$$

$$\text{C}\ \frac{A_1, \ldots, A_n \Rightarrow B \qquad B \Rightarrow A_1 \quad \ldots \quad B \Rightarrow A_n}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B, \Delta} \qquad \text{MC}\ \frac{A_1, \ldots, A_n \Rightarrow B}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B, \Delta}$$

But:

- Not modular.
- Modal rules are not invertible:
  - A single failed proof doesn't imply non-derivability.
  - Need of backtracking.



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$$\text{C}\ \frac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1 \quad ... \quad B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta} \qquad \text{MC}\ \frac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B, \Delta}$$

But:

- Not modular.
- Modal rules are not invertible
- Complex countermodel extraction for non-valid formulas (Lavendhomme & Lucas 2000):
  - Needs to keep track of all possible applications of modal rules.
  - Makes use of analytic cut:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Further proof systems for non-normal modal logics

1957	Ohnishi, Matsumoto	Gentzen calculus for **MCT**
1983	Fitting	Prefixed tableaux for **M**
1999	Governatori, Luppi	Labelled tableaux for monotonic logics
2000	Lavendhomme, Lucas	Gentzen calculi for the classical cube
2005/2011	Indrzejczak	Gentzen calculi for extensions of the classical cube
2007	Indrzejczak	Prefixed tableaux calculi
2014/2020	Orlandelli	Gentzen calculi for *P*, *D*
2015	Gilbert, Maffezioli	Labelled sequent calculi based on multi-relational semantics
2015/2019	Lellmann, Pimentel	Nested/linear nested sequent calculi
2017	Negri	Labelled sequent calculi based on neighbourhood semantics
2018	D., Olivetti, Negri	Labelled sequent calculi based on bi-neighbourhood semantics
2019	Chen et al.	Display calculi for monotonic logics

A hypersequent calculus
for non-normal modal logics

Hypersequents with blocks (D., Lellmann, Olivetti, Pimentel 2021)

Sequent calculi extended with additional structural connectives

Block: $\langle \Sigma \rangle$, where Σ multiset of formulas.

Sequent: $\Gamma, \langle \Sigma_1 \rangle, ..., \langle \Sigma_n \rangle \Rightarrow \Delta$.

Hypersequent: $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$.

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Formula interpretation

- $\iota(\langle A_1, ..., A_n \rangle) = \Box(A_1 \wedge ... \wedge A_n)$.
- $\iota(\Gamma, \langle \Sigma_1 \rangle, ..., \langle \Sigma_m \rangle \Rightarrow \Delta) = \bigwedge \Gamma \wedge \bigwedge_{j \leq m} \Box \bigwedge \Sigma_j \rightarrow \bigvee \Delta$.
- No formula interpretation for hypersequents.

Semantic interpretation

- $w \Vdash \Gamma \Rightarrow \Delta$ iff $w \Vdash \iota(\Gamma \Rightarrow \Delta)$.
- $\mathcal{M} \models \Gamma \Rightarrow \Delta$ iff $w \Vdash \Gamma \Rightarrow \Delta$ for every w of $\mathcal{M}$.
- $\mathcal{M} \models \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ iff $\mathcal{M} \models \Gamma_i \Rightarrow \Delta_i$ for some $i \in \{1, ..., n\}$.

Hypersequents with blocks (D., Lellmann, Olivetti, Pimentel 2021)

Sequent calculi extended with additional structural connectives

Block: $\langle \Sigma \rangle$, where Σ multiset of formulas.

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Hypersequent: $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$.

Advantage of blocks

- Modular definition of the calculi.
- Extensions simply defined by rules handling blocks.

Advantage of hypersequents

- All rules are invertible.
- Decision procedure by a single proof.

Hypersequent calculi $\mathbf{H.E}^*$

Propositional rules (examples)

$$\mathsf{L}\wedge\ \frac{\mathcal{G} \mid \Gamma, A \wedge B, A, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\mathcal{G} \mid \Gamma \Rightarrow A, A \wedge B, \Delta \qquad \mathcal{G} \mid \Gamma \Rightarrow B, A \wedge B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B, \Delta}$$

Modal rules for the classical cube

$$\mathsf{L}\Box\ \frac{\mathcal{G} \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{R}\Box\mathsf{m}\ \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$$\mathsf{R}\Box\ \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \qquad \{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$$\mathsf{N}\ \frac{\mathcal{G} \mid \Gamma, \langle \top \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{C}\ \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

Modal rules for further extensions (examples)

$$\mathsf{T}\ \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta} \qquad \mathsf{P}\ \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}$$

Semantic intuition

- Components represent the worlds of a model.
- Blocks represent truth sets belonging to the neighbourhood: $\langle A \rangle \approx [\![A]\!] \in \mathcal{N}(w)$.
- Rules express semantic conditions. Examples:

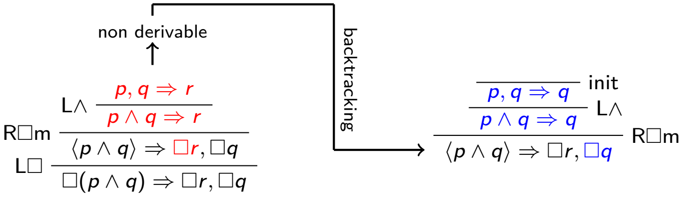
$$\text{L}\Box \; \frac{\mathcal{G} \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta} \qquad w \Vdash \Box A \Longrightarrow [\![A]\!] \in \mathcal{N}(w).$$

$$\text{N} \; \frac{\mathcal{G} \mid \Gamma, \langle \top \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad [\![\top]\!] = \mathcal{W} \in \mathcal{N}(w).$$

$$\text{C} \; \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta} \qquad [\![\bigwedge \Sigma]\!], [\![\bigwedge \Pi]\!] \in \mathcal{N}(w) \Longrightarrow [\![\bigwedge \Sigma]\!] \cap [\![\bigwedge \Pi]\!] \in \mathcal{N}(w).$$

- Cumulative rules: A saturated hypersequent contains all information to build a countermodel.

Without hypersequents



With hypersequents

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle p\wedge q\rangle \Rightarrow \Box r,\Box q \mid p,q\Rightarrow r \mid p,q\Rightarrow q}{\langle p\wedge q\rangle \Rightarrow \Box r,\Box q \mid p,q\Rightarrow r \mid p\wedge q\Rightarrow q}\;\text{L}\wedge}{\langle p\wedge q\rangle \Rightarrow \Box r,\Box q \mid p,q\Rightarrow r}\;\text{R}\Box\text{m}}{\langle p\wedge q\rangle \Rightarrow \Box r,\Box q \mid p\wedge q\Rightarrow r}\;\text{L}\wedge}{\langle p\wedge q\rangle \Rightarrow \Box r,\Box q}\;\text{R}\Box\text{m}}{\Box(p\wedge q) \Rightarrow \Box r,\Box q}\;\text{L}\Box$$

Properties of **H.E***

- Syntactic proof of admissibility of structural rules and cut

$$\text{Lwk}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta} \quad \text{Rwk}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta} \quad \text{Ewk}\ \frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \quad \text{Lctr}\ \frac{\mathcal{G} \mid \Gamma, \phi, \phi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta}$$

$$\text{Lctr}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow A, A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta} \quad \text{Ectr}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \quad \text{Bctr}\ \frac{\mathcal{G} \mid \Gamma, \langle \Theta, A, A \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Theta, A \rangle \Rightarrow \Delta}$$

$$\text{Bmgl}\ \frac{\mathcal{G} \mid \Gamma, \langle \Theta, A \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Theta, A, A \rangle \Rightarrow \Delta} \quad \text{cut}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta \qquad \mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

$$\text{sub}\ \frac{\mathcal{G} \mid \Sigma \Rightarrow A \qquad \{\mathcal{G} \mid A \Rightarrow B\}_{B \in \Sigma} \qquad \mathcal{G} \mid \Gamma, \langle A \rangle \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}$$

- Soundness and completeness w.r.t. **E*** (via simulation of sequent calculi)

We now consider

- Countermodel extraction
- Complexity of proof search

Countermodel extraction from saturated hypersequent

- Every component corresponds to a world.
- Formulas in Γ are true, formulas in Δ are false.

Countermodel extraction from saturated hypersequent

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$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_3 \Rightarrow \Delta_3 \mid \Gamma_4 \Rightarrow \Delta_4 \mid \Gamma_5 \Rightarrow \Delta_5 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

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$$A \in \Gamma_i$$

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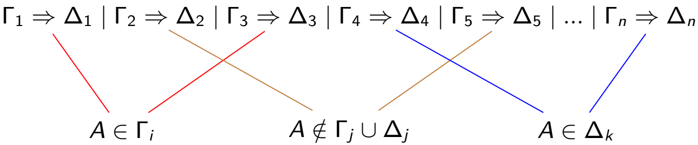

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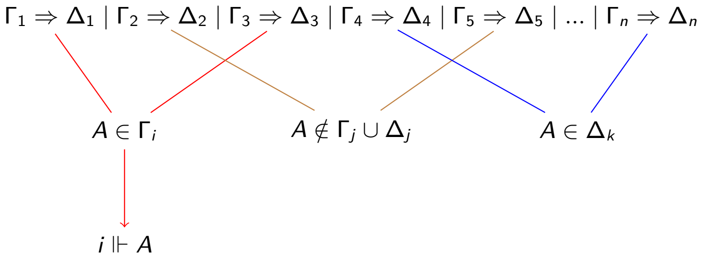

Countermodel extraction from saturated hypersequent

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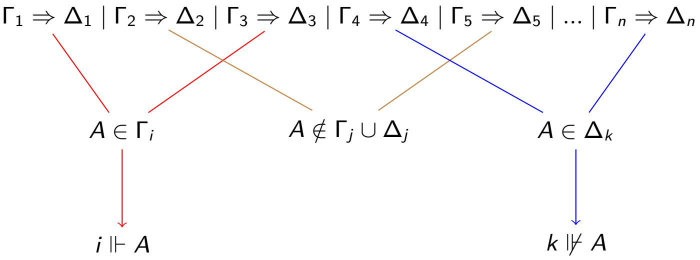

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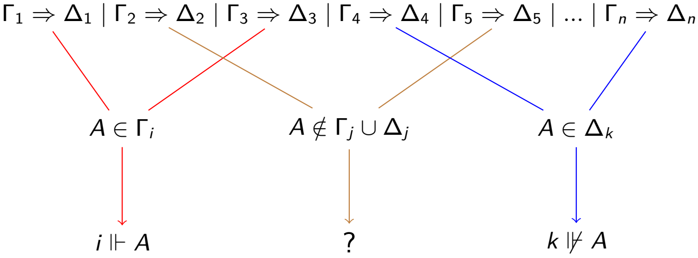

Countermodel extraction from saturated hypersequent

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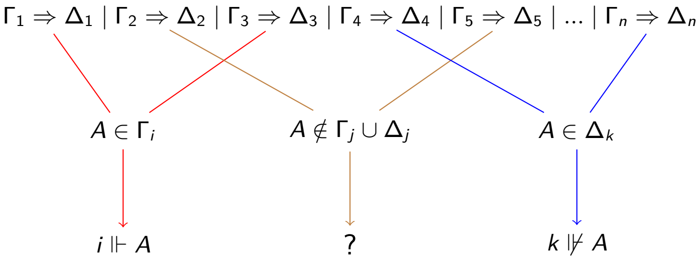


Impossible to determine $[\![A]\!]$.

$\Rightarrow$ Impossible to define directly a standard model.

Countermodel extraction from saturated hypersequent

- Every component corresponds to a world.
- Formulas in Γ are true, formulas in Δ are false.

1st solution

Saturate with analytic cut:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{G} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{ cut}$$

Pros

- Fixes the extension of every subformula
- Constructs a standard neighbourhood model

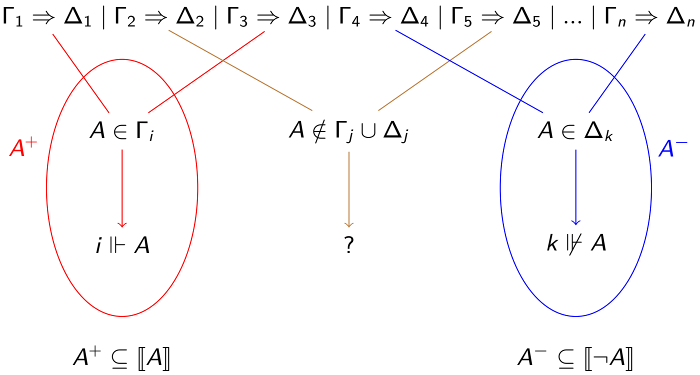
Cons

- Strong increase in complexity

Countermodel extraction from saturated hypersequent

- Every component corresponds to a world.
- Formulas in Γ are true, formulas in Δ are false.

2nd solution



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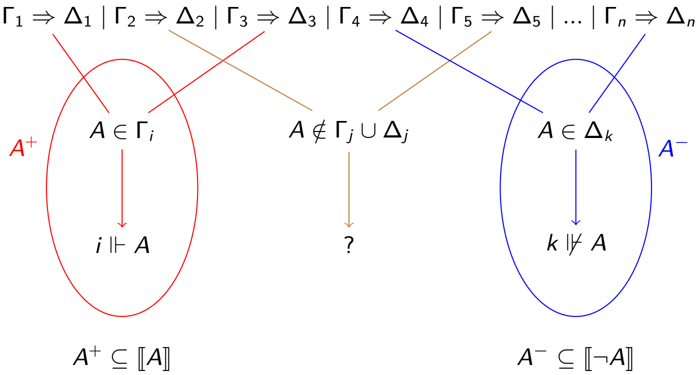
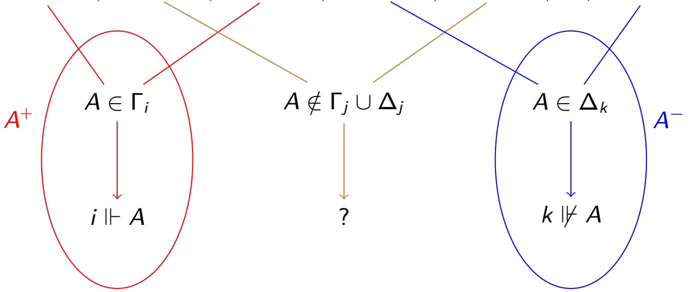


Bi-neighbourhood semantics!

- Every component corresponds to a world.
- Formulas in Γ are true, formulas in Δ are false.

2nd solution

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_3 \Rightarrow \Delta_3 \mid \Gamma_4 \Rightarrow \Delta_4 \mid \Gamma_5 \Rightarrow \Delta_5 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$$



$A^+ \subseteq [\![A]\!]$ $\qquad$ $A^- \subseteq [\![\neg A]\!]$

$$\langle A \rangle \in \Gamma_m \quad \rightarrow \quad (A^+, A^-) \in \mathcal{N}(m)$$

Example of countermodel

Open branch of failed proof of K in **H.EC**

saturated

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q, p}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q}\ \mathrm{L}\!\to}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p}\ \mathrm{R}\Box}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q}\ \mathrm{R}\Box}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle \Rightarrow \Box q}\ \mathrm{C}}{\Box(p \to q), \Box p \Rightarrow \Box q}\ \mathrm{L}\Box^2
$$

Example of countermodel

Open branch of failed proof of K in **H.EC**

$$
\mathcal{W} = \qquad \begin{array}{c}
\hspace{6em} 1 \hspace{10em} 2 \hspace{7em} 3 \\
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q, p}{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q} \; \mathrm{L}\!\to}{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q \mid q \Rightarrow p} \; \mathrm{R}\Box}{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q} \; \mathrm{R}\Box}{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle \Rightarrow \Box q} \; \mathrm{C}}{\Box(p \to q), \Box p \Rightarrow \Box q} \; \mathrm{L}\Box^2
\end{array}
$$

Example of countermodel

Open branch of failed proof of K in **H.EC**

$$
\mathcal{W} = \quad
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \overset{1}{\Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q} \mid \overset{2}{q \Rightarrow p} \mid \overset{3}{p \to q \Rightarrow q, p}
        }{
          \Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q
        }\ \mathrm{L}{\to}
      }{
        \Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q \mid q \Rightarrow p
      }\ \mathrm{R}\Box
    }{
      \Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q
    }\ \mathrm{R}\Box
  }{
    \Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle \Rightarrow \Box q
  }\ \mathrm{C}
}{
  \Box(p \to q), \Box p \Rightarrow \Box q
}\ \mathrm{L}\Box^2
$$

Bi-neighbourhood countermodel

- $\mathcal{W} = \{1, 2, 3\}$.
- $$\begin{aligned} \mathcal{N}(1) \quad = \quad & \{(p \to q^{+}; p \to q^{-}), & = \quad & \{(\{3\}, \emptyset), \\ & (p^{+}; p^{-}), & & (\emptyset, \{2,3\}), \\ & (p \to q, p^{+}; p \to q, p^{-})\} & & (\emptyset, \{2,3\})\}. \end{aligned}$$
- $\mathcal{N}(2) = \mathcal{N}(3) = \emptyset$.
- $\mathcal{V}(p) = \emptyset, \mathcal{V}(q) = \{2\}$.

Proof search strategy for a hypersequent $\mathcal{H}$

- Bottom-up proof search with simple redundancy check: a rule is not applied if the premiss is already contained in the hypersequent (needed because rules are cumulative).
- Complexity upper bound for NNMLs determined by the cost of construction of a proof search tree $\mathcal{T}$ for $\Rightarrow A$.

Complexity for the logics without axiom C

- Maximal length of a hypersequent occurring in $\mathcal{T}$ is polynomial (wrt $\mathcal{H}$) (by counting the possible formulas, blocks and components in it).
- Maximal length of a branch of $\mathcal{T}$ is polynomial.
- Cost of redundancy check is polynomial.

$\Rightarrow$ The hypersequent calculi provide a coNP decision procedure for the derivability problem.

Vs. complexity for the logics with axiom C

- n $\Box$-subformulas $\Rightarrow$ 2^n blocks.
- $\mathcal{T}$ can contain hypersequents of exponential size wrt $\mathcal{H}$.

$\Rightarrow$ Sub-optimal proof-search.

From syntactical to semantical properties

Logics without axiom C

1. Complexity of proof search + direct countermodel extraction

$\Rightarrow$ NP decision procedure for the satisfiability problem (proof search for $A \Rightarrow$)

2. Polynomial size of saturated hypersequents + 1-1 correspondence between components-worlds and blocks-neighbourhood sets

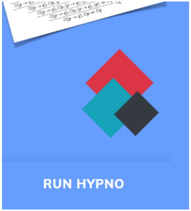
$\Rightarrow$ Polysize model property for bi-neighbourhood semantics (constructive proof): every satisfiable formula has a model of polynomial size (counting both $\mathcal{W}$ and $\mathcal{N}$).

3. Indirect polysize model property for neighbourhood semantics, considering a trasformation from bi-neighbourhood to standard models.

From semantical to negative syntactical properties, for logics with C

Conjecture: Satisfiable formulas of size n whose models have at least 2^n worlds. Known for **K** (Blackburn et al. 2001).

$\Rightarrow$ No PSPACE proof search procedure is possible that explicitly constructs a countermodel.



NONNORMAL MODAL LOGIC: ○ E ○ EM ○ EN ○ EC ○ EMN ◉ EMC ○ ENC ○ EMNC

(box(a ^ b)) -> (box(a))

run HYPNO

RESULT: VALID

BUILD A DERIVATION

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, b, a \wedge b \Rightarrow a \mid \langle a \wedge b\rangle, \Box(a \wedge b) \Rightarrow \Box(a), \Box(a \wedge b) \to \Box(a)}\ \textbf{init}}{a \wedge b \Rightarrow a \mid \langle a \wedge b\rangle, \Box(a \wedge b) \Rightarrow \Box(a), \Box(a \wedge b) \to \Box(a)}\ \wedge_L}{\langle a \wedge b\rangle, \Box(a \wedge b) \Rightarrow \Box(a), \Box(a \wedge b) \to \Box(a)}\ M\Box_R}{\Box(a \wedge b) \Rightarrow \Box(a), \Box(a \wedge b) \to \Box(a)}\ \Box_L}{\Rightarrow \Box(a \wedge b) \to \Box(a)}\ \to_R
$$

RESULT: NOT VALID

BUILD A COUNTER-MODEL

COUNTERMODEL FOR ((BOX(A)) ^ (BOX(B))) -> (BOX(A ^ B))

W = {3, 2, 1}

V(a)={3}, V(b)={2}
For all other atomic variables P (if any), V(P)=∅

N (NON-MONOTONIC CASE):

N(3)=∅
N(2)=∅
N(1)={([2],[3]),([3],[2])}

`http://193.51.60.97:8000/HYPNO/`

Main reference so far:

TD, B. Lellmann, N. Olivetti, E. Pimentel.
Hypersequent calculi for non-normal modal and deontic logics: Countermodels and optimal complexity. JLC 2021.

More recently, application of these methods/results to

- Agency logics of "bringing-it-about-that" (BIAT logics)
  (with Charles Grellois and Nicola Olivetti)
- More expressive languages: modal description logics
  (with Andrea Mazzullo, Ana Ozaki and Nicolas Troquard)
- Combinations of NNMLs
  (work in progress with Andrea Mazzullo)

Combinations of NNMLs

(work in progress)

Combinations of NNMLs (I)

- Hypersequent rules are modular and context independent.

⇒ Can be immedately extended to multimodal setting.

Fusions (axiomatic definition)

Given NNMLs $\mathbf{L}_1, ..., \mathbf{L}_n$ in $\mathcal{L}_{\Box_1}, ..., \mathcal{L}_{\Box_n}$ (sharing atoms and boolean connectives), their fusion $\mathbf{L}_1 \oplus ... \oplus \mathbf{L}_n$ is the smallest logic in $\mathcal{L}_{\Box_1 ... \Box_n}$ containing $\mathbf{L}_1 \cup ... \cup \mathbf{L}_n$ and closed under the rules of $\mathbf{L}_1$, ..., $\mathbf{L}_n$.

Semantics for fusions $\mathbf{L}_1 \oplus ... \oplus \mathbf{L}_n$

n-neighbourhood models $\langle \mathcal{W}, \mathcal{N}_1, ..., \mathcal{N}_n, \mathcal{V} \rangle$, where each $\mathcal{N}_i$ satisfies the conditions for $\mathbf{L}_i$.

- Note: independently axiomatizable logics. Not true for all NNMLs (Fajardo & Finger 2005).
- General results for normal Ls (Wolter 1998), NNML fusions less studied.
- From the hypersequent calculus we obtain:
  - Proof search and countermodel extraction extended to fusions.
  - w/o axiom C, proof search is still (co)NP.

  ⇒ Fusions of NP NNMLs are NP (vs. e.g. **S5**).

Back to formula interpretation

- No formula interpretation for hypersequents.
- Semantically, a hypersequent is a disjunction of validities:
  $\mathcal{M} \models \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ iff $\mathcal{M} \models \Gamma_i \Rightarrow \Delta_i$ for some $i \in \{1, \dots, n\}$.
- Non-normal modalities are not strong enough.

We add a universal modality

- $\mathcal{M}, w \Vdash \mathcal{U}A$ iff for all v, $\mathcal{M}, v \Vdash A$.
- $\iota(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) = \mathcal{U}(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \dots \vee \mathcal{U}(\bigwedge \Gamma_n \to \bigvee \Delta_n)$.
- We add hypersequent rules for **S5** (Restall 2005, Poggiolesi 2008):

$$\mathsf{L}\mathcal{U}\ \frac{\mathcal{H} \mid \Gamma, \mathcal{U}A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{H} \mid \Gamma, \mathcal{U}A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \mathsf{R}\mathcal{U}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \mathcal{U}A, \Delta \mid \ \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \mathcal{U}A, \Delta}$$

$$\mathcal{U}_{\mathrm{t}}\ \frac{\mathcal{H} \mid \Gamma, \mathcal{U}A, A \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \mathcal{U}A \Rightarrow \Delta}$$

- Fusions of logics also sharing the symbol $\mathcal{U}$.
- n-neighbourhood models, with $w \Vdash \mathcal{U}A$ iff $v \Vdash A$ for all v.

- Proof search is still NP (w/o axiom C)
- Same countermodel extraction, gives models with $\mathcal{U}$ universal modality.

⇒ Completeness of the calculus wrt n-neighbourhood models.

⇒ NP-completeness of satisfiability problem (w/o axiom C).

- Equivalent axiomatic systems (via full formula interpretation).
- Additional axioms (examples):

$$E_i^{\mathcal{U}} \quad \mathcal{U}(A \to B) \wedge \mathcal{U}(B \to A) \to \mathcal{U}(\Box_i A \to \Box_i B) \qquad N_i^{\mathcal{U}} \quad \mathcal{U}A \to \mathcal{U}\Box_i A$$
$$M_i^{\mathcal{U}} \quad \mathcal{U}(A \to B) \to \mathcal{U}(\Box_i A \to \Box_i B) \qquad P_i^{\mathcal{U}} \quad \mathcal{U}\neg A \to \mathcal{U}\neg\Box_i A$$

⇒ Indirect completeness of axiom systems wrt n-neighbourhood models.

Note: Almost immediate, no need of (e.g.) generated submodels.

BIAT agency logics

Basic priciples of BIAT

- Actions as results, means do not matter.
- Focus on responsibility, e.g. $\neg\mathbb{E}\top$.

Two modalities: Does & Can, indexed by agents

- $\mathbb{E}_i A$ "Agent i b.i.a.t. A".
- $\mathbb{C}_i A$ "Agent i is capable of b.i.a.t. A".

BIAT axioms (Elgesem 1997)

- Principle of success: $(T_{\mathbb{E}})$ $\mathbb{E}_i A \rightarrow A$
- Principle of aggregation: $(C_{\mathbb{E}})$ $\mathbb{E}_i A \wedge \mathbb{E}_i B \rightarrow \mathbb{E}_i(A \wedge B)$
- Do implies Can: $(Int_{\mathbb{E}\mathbb{C}})$ $\mathbb{E}_i A \rightarrow \mathbb{C}_i A$
- Principle of possibility: $(P_{\mathbb{C}})$ $\neg\mathbb{C}_i\bot$
- Principle of avoidability: $(Q_{\mathbb{C}})$ $\neg\mathbb{C}_i\top$
- Actions are not sensitive to their syntactic formulation:

$$(RE_{\mathbb{E}})\ \frac{A \leftrightarrow B}{\mathbb{E}_i A \leftrightarrow \mathbb{E}_i B} \qquad (RE_{\mathbb{C}})\ \frac{A \leftrightarrow B}{\mathbb{C}_i A \leftrightarrow \mathbb{C}_i B}$$

BIAT agency logics

Strictly non-normal

- No monotonicity: $\models A \rightarrow B \;\not\Rightarrow\; \models \mathbb{E}_i A \rightarrow \mathbb{E}_i B$ (otherwise $\mathbb{E}_i A \rightarrow \mathbb{E}_i \top$)
- No necessitation: $\models A \;\not\Rightarrow\; \models \mathbb{E}_i A$ (otherwise $\mathbb{E}_i \top$)

$\Rightarrow$ Incompatibile with normal modalities.

  - Contains the negation of necessitation: $\models \neg \mathbb{E}_i \top$
  - No normal extension is possible.

Semantics

- Selection function models (Elgesem 1997)
- Neighbourhood models (Governatori & Rotolo 2005)
- Bi-neighbourhood models

Extensions

- With attempted actions (Jones & Parent 2007), time (Troquard 2019), coalitions (Troquard 2014), actions by means/dyadic modalities (McNamara 2019).
- We give calculus for basic logic, coalitions and dyadic modalities (D., Grellois, Olivetti 2023).

Back to the calculus

- Propositional rules and rules for each modality (as before)
- Rules for interaction (example)

$$\text{Int}_{\mathbb{E}\mathbb{C}} \ \frac{\mathcal{G} \mid \Gamma, \langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle\Sigma\rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

Back to the calculus

- Propositional rules and rules for each modality (as before)
- Rules for interaction

- Blocks allow analytic rules for the relation between BIAT modalities.
- By modularity:
  - Rules for ifferent BIAT modalities independend from each other.
  - Easy definition of calculi for extensions.
  - e.g. rules of basic calculus + rules for coalitions (examples):

$$\mathsf{F}_{\mathbb{C}} \frac{}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_{\emptyset} \Rightarrow \Delta} \qquad \mathsf{Int}^2_{\mathbb{E}\mathbb{C}} \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_{g_1}, \langle \Pi \rangle^{\mathbb{E}}_{g_2}, \langle \Sigma, \Pi \rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_{g_1}, \langle \Pi \rangle^{\mathbb{E}}_{g_2} \Rightarrow \Delta}$$

Back to the calculus

- Propositional rules and rules for each modality (as before)
- Rules for interaction

- Blocks allow analytic rules for the relation between BIAT modalities.
- Modularity

- Termination of proof search and countermodel extraction

⇒ Decidability of satisfiability problem for BIAT logics

Note: exponential models because of the rule

$$\mathsf{C}_{\mathbb{E}} \; \frac{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}}, \langle \Sigma, \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

Conclusions

- Good match of bi-neighbourhood semantics and partial information given by hypersequent proof search.
- From syntactical to semantical properties:
  - Complexity of the satisfiability problem.
  - Polysize model property.
- Also in contrapositive way:
  - Exponential models for C $\rightsquigarrow$ No optimal calculus giving countermodels.
- Application to fusions, combinations with universal modality, BIAT logics.

Main open problem

- Iterative axioms, e.g. 4, 5, B (no cut-free calculus at all in some cases).

Conclusions

- Good match of bi-neighbourhood semantics and partial information given by hypersequent proof search.
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Thank you!

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