Value groups of finitely ramified henselian valued fields and model completeness

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Valued fields

Let K be a field and G a totally ordered abelian group. A *valuation* over K is a surjective map

$$v : K \longrightarrow G \cup \{\infty\}$$

satisfying the following conditions:

- $v(a) = \infty \iff a = 0$;
- $v(ab) = v(a) + v(b)$;
- $v(a+b) \geq min\{v(a), v(b)\}$.

$O = \{x \in K \mid v(x) \geq 0\}$ is the valuation ring. It is a local ring with unique maximal ideal $M = \{x \in K \mid v(x) > 0\}$. The quotient $O/M = k$ is the *residue field* of the valuation v.

Cases:

- **1.** $char\, K = char\, k = 0$;
- **2.** $char\, K = 0$, $char\, k = p$;
- **3.** $char\, K = char\, k = p$.

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- **3.** $char\, K = char\, k = p$.

Examples

1. Let p be a prime. The field of p-adic numbers

$$\mathbb{Q}_p = \left\{ a = \sum_{i \geq k}^{\infty} a_i p^i \,|\, k \in \mathbb{Z}, a_i \in \{0, \ldots, p-1\} \right\}$$

with the valuation $v_p(a) = min\{i \,|\, a_i \neq 0\}$ (the p-adic valuation) with values in $\mathbb{Z}$ and residue field $\mathbb{F}_p$.

2. Let k be a field and G an ordered abelian group. The valued field of *generalized power series* (or *Hahn field*)

$$k((G)) = \left\{ a = \sum_{g \in G} a_g t^g \,|\, a_g \in k, \text{ for all } g \in G \text{ and } supp(a) \text{ is well ordered} \right\}$$

with the valuation $v_t(a) = min\{g \,|\, a_g \neq 0\}$ (the t-adic valuation) with values in G and residue field k.

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$$\mathbb{Q}_p = \left\{ a = \sum_{i \geq k}^{\infty} a_i p^i \mid k \in \mathbb{Z}\,, a_i \in \{0, \dots, p-1\} \right\}$$

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Valued fields

Definition

A valued field is henselian if its valuation extends uniquely to every algebraic extension.

Remark. $(\mathbb{Q}_p, v_p)$ is henselian.

Theorem (Hensel's Lemma)

Suppose K is a pseudo-complete valued field, then it is henselian.

Definition

Let K be a valued field. We say that K is finitely ramified if $char(k) = p$ and $|\{v(x) : 0 < v(x) \leq v(p)\}| = e < \infty$. In particular, if $e = 1$ the field is unramified. The element with minimal positive valuation is called uniformizer.

AKE - equicharacteristic zero

Consider $\mathfrak{L}_{rings} = \{+, \cdot, 0, 1\}$, $\mathfrak{L}_{oags} = \{+, 0, \leq\}$, and $\mathfrak{L}_{vf} = (\mathfrak{L}_{Rings}, \mathfrak{L}_{oags}, \mathfrak{L}_{rings}, v, res)$.

Theorem (Ax-Kochen/Ershov principle)

Let $(K_1, v_1), (K_2, v_2)$ be two henselian valued field whose residue fields k_1, k_2 have characteristic 0 and let G_1, G_2 be their value groups. Then

$$K_1 \equiv_{vf} K_2 \text{ iff } k_1 \equiv_{rings} k_2 \text{ and } G_1 \equiv_{oag} G_2$$

Theorem (Ax-Kochen/Ershov principle ($\preceq$-version))

Let $(K_1, v_1), (K_2, v_2)$ be two henselian valued field whose residue fields k_1, k_2 have characteristic 0 and let G_1, G_2 be their value groups. Then

$$K_1 \preceq_{vf} K_2 \text{ iff } k_1 \preceq_{rings} k_2 \text{ and } G_1 \preceq_{oag} G_2$$

AKE - mixed characteristic

Theorem (Bélair)

Let $(K_1, v_1), (K_2, v_2)$ be two unramified henselian valued field with perfect residue fields k_1, k_2 and let G_1, G_2 be their value groups. Then

$$K_1 \preceq_{vf} K_2 \text{ iff } k_1 \preceq_{rings} k_2 \text{ and } G_1 \preceq_{oag} G_2$$

Theorem (Anscombe-Jahnke)

Let $(K_1, v_1), (K_2, v_2)$ be two unramified henselian valued field with arbitrary residue fields k_1, k_2 and let G_1, G_2 be their value groups. Then

$$K_1 \preceq_{vf} K_2 \text{ iff } k_1 \preceq_{rings} k_2 \text{ and } G_1 \preceq_{oag} G_2$$

Question: Does the transfer hold for valued fields with (finite) ramification? **Answer**: In general, no.

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Z-groups

Definition

An ordered abelian group is called a $\mathbb{Z}$-group if it is elementarily equivalent to $\mathbb{Z}$ as an ordered abelian group.

Proposition

The theory of $\mathbb{Z}$ as an ordered abelian group has quantifier elimination in the Presburger language $\mathfrak{L}_{pres} = \{+, 0, 1, \leq, \equiv_m\}_{m \in \mathbb{N}}$, where 1 is a constant for the minimal positive element and $a \equiv_m b$ iff $a - b \in m\mathbb{Z}$.

Theorem (Derakhshan-Macintyre)

Let K be a Henselian valued field of mixed characteristic $(0, p)$ with finite ramification $e \geq 1$. Suppose the value group of K is a $\mathbb{Z}$-group. If the theory of the residue field k is model complete in the language of rings, then the theory of K is model complete in the language of rings.

Some observations

(i) let k be a field. If $Th(k)$ is model complete in $\mathfrak{L}_{rings}$, then k is perfect.

(ii) Let K be an henselian valued field of mixed characteristic $(0, p)$ and ramification index e, and let $n > e$ be an integer coprime with p. Then the valuation ring is existentially definable by the formula

$$\exists y(1 + px^n = y^n),$$

and the maximal ideal is existentially definable by the formula

$$\exists y(1 + \frac{1}{p}x^n = y^n).$$

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Oags with finite spines

Let G be an oag. For each positive integer n we recall the definition of the spine S_n:

Definition

For $n \in \mathbb{N}$ and $a \in G \setminus nG$, let H_a be the largest convex subgroup of G such that $a \notin H_a + nG$; set $H_a = 0$ if $a \in nG$. Define $S_n := G/\sim$, with $a \sim a'$ iff $H_a = H_{a'}$. and let $s_n : G \longrightarrow S_n$ be the canonical projection. For $\alpha = s_n(a) \in S_n$, define $\overline{H_\alpha} := H_a$. Since the system of convex subgroups of an ordered abelian group are linearly ordered, S_n is an interpretable set linearly ordered by $\alpha \leq \alpha'$ if $\overline{H_\alpha} \subseteq \overline{H_{\alpha'}}$. The structure $(S_n, <)$ is called the n-spine of G.

Oags with finite spines

Definition

If G is an oag such that for each $n \in \mathbb{N}$, $|S_n|$ is finite, G is said to have finite spines.

Remark. All the $\overline{H_\alpha}$ are definable in $\mathfrak{L}_{oag}$ and, moreover, if G is a group with finite spines, then $\{\overline{H_\alpha} \mid \alpha \in S_n,\ n \in \mathbb{N}\}$ are all the definable convex subgroups of G.
$\Longrightarrow$ G has only finitely or countably many definable convex subgroups that we denote with $(H_i)_{i \in I}$, where I is a finite or countable set of indexes.

QE

Proposition (Halevi-Hasson/Farré)

Let G be an ordered abelian group with finite spines and let $\{H_i\}_{i\in I}$ *be its definable convex subgroups for some I finite or countable. Then G has quantifier elimination in the the language:*

$$\mathfrak{L} = \mathfrak{L}_{oag} \cup \{(x =_{H_i} y + j1_i)_{i\in I, j\in\mathbb{Z}}, (x \equiv_m^{H_i} y + j1_i)_{i\in I, j\in\mathbb{Z}, m\in\mathbb{N}}\}$$

where $j1_i$ *is j times a representative* $1_i \in G$ *of the minimal positive element of the quotient* G/H_i*, if it exists,* 0 *otherwise.*

Model completeness

Proposition (D.M.)

Let G be an ordered abelian group with finite spines and let $(H_i)_{i \in I}$ be its definable convex subgroups. Then G is model complete in the language:

$$\mathfrak{L}^*_{oag} = \{0, +, -, \leq, (j1_i + H_i)_{j=0,1; i \in I}\},$$

where $1_i \in G$ is a representative of the minimal element of the quotient G/H_i if it is discrete, 0 otherwise.

Model completeness with value group with finite spines

Theorem (D.M.)

Let K be an Henselian valued field of mixed characteristic $(0, p)$, finite ramification $e \geq 1$, and value group G with finite spines. If the theory of the residue field k is model complete in the language of rings, then the theory of K is model complete in the language $\mathfrak{L}^*_{Ring} = \{0, +, \cdot, 1, A_{i,j}\}_{j=0,1;i\in I}$ where $A_{i,j}$ is a predicate such that

$$A^K_{i,j} = \{a \in K | v(a) = j1^G_i \mod H_i\},$$

where the $(H_i)_{i\in I}$ are the definable convex subgroups of G and $1_i \in G$ is a representative for the minimal element of the quotient G/H_i if it is discrete, 0 otherwise.

Proof

Let $(K_1, v_1), (K_2, v_2) \models Th(K)$ (assume $\aleph_1$-saturation), such that $K_1 \subseteq K_2$ in $\mathfrak{L}^*_{Ring}$.
Claim: $(K_1, v_1) \preceq (K_2, v_2)$ in $\mathfrak{L}^*_{Rings}$. Note that:

- let Δ be the minimal convex subgroup of G. Then $Th(G/\Delta)$ is model complete in $\mathfrak{L}^*_{oag}$;
- $k_1 \preceq k_2$ implies $\mathring{K}_1 \preceq \mathring{K}_2$;
- AKE in the equicharacteristic 0 case obtained by coarsening holds resplendently considering an expansion of the language of groups;
- The languages $(\mathfrak{L}^*_{Ring}, \mathfrak{L}_{oag}, \mathfrak{L}_{ring}, v, res)$ and $(\mathfrak{L}_{Ring}, \mathfrak{L}^*_{oag}, \mathfrak{L}_{ring}, v, res)$ are bi-interpretable.
- the valuation is still $\exists$ and $\forall$-definable by the same formula;

Example. Hahn series in one variable

Consider the field of Hahn series $\mathbb{Q}_p((t^{\mathbb{Z}}))$ and the valuation

$$val : \mathbb{Q}_p((t^{\mathbb{Z}})) \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

such that

$$O_{val} = \{x | val(x) \geq 0\} = \{x | v_t(x) > 0 \text{ or } v_t(x) = 0 \wedge v_p(ac_t(x)) \geq 0\}.$$

By the Theorem, $Th((\mathbb{Q}_p((t^{\mathbb{Z}})), val))$ is model complete in the language of rings together with two predicates A_0, A_1 such that

$$\begin{aligned} A_0^{\mathbb{Q}_p((t^{\mathbb{Z}}))} &= \{x \,|\, val(x) \in \{0\} + \mathbb{Z}\} \\ &= \left\{x \,|\, x = \textstyle\sum_{i \geq 0} a_i t^i, a_0 \neq 0\right\} = \mathbb{Q}_p + (t)^{>0}. \end{aligned}$$

$$\begin{aligned} A_1^{\mathbb{Q}_p((t^{\mathbb{Z}}))} &= \{x \,|\, val(x) = (1,0) \mod (\{0\} + \mathbb{Z})\} \\ &= \left\{x \,|\, x = \textstyle\sum_{i \geq 1} a_i t^i, a_1 \neq 0\right\} = (t)^{>0}. \end{aligned}$$

Example. Hahn series with many variables

We can consider the valuation

$$val_n : \mathbb{Q}_p((t_1^{\mathbb{Z}}))\dots((t_n^{\mathbb{Z}})) \longrightarrow \bigoplus_{i=1}^{n+1} \mathbb{Z}$$

such that

$$O_{val_n} = \bigcup_{i=0}^{n} O^i,$$

where

$$\begin{aligned} O^n &= \{x \mid v_{t_n}(x) > 0\} \\ O^{n-1} &= \{x \mid v_{t_n}(x) = 0 \wedge v_{t_{n-1}}(ac_{t_n}(x)) > 0\} \\ &\vdots \\ O^0 &= \{x \mid v_{t_n}(x) = 0 \wedge v_{t_{n-1}}(ac_{t_n}(x)) = 0 \wedge \dots \\ &\wedge v_{t_1}(ac_{t_2}(\dots(ac_{t_n}(x)))) = 0 \wedge v_p(ac_{t_1}(\dots(ac_{t_n}(x))..)) > 0\} \end{aligned}$$

Example. Hahn series with many variables

By the Theorem, the theory of the valued field $\mathcal{K} = (\mathbb{Q}_p((t_1^{\mathbb{Z}}))\ldots((t_n^{\mathbb{Z}})), val_n)$ is model complete in the language of rings together with predicates $A_{i,j}$, $i = 1, \ldots, n$, $j = 0, 1$, such that

$$A_{i,0}^{\mathcal{K}} = \{x \in \mathcal{K} \mid val_n(x) \in H_i\} = \{x \in \mathcal{K} \mid x \in O^i \wedge v_{t_i}(ac_{t_{i+1}}(\ldots(ac_{t_n}(x))..)) = 0\}.$$

$$A_{i,1}^{\mathcal{K}} = \{x \in \mathcal{K} \mid val_n(x) = c_i^G \mod H_i\} = \{x \in \mathcal{K} \mid x \in O^i \wedge v_{t_i}(ac_{t_{i+1}}(\ldots(ac_{t_n}(x))..)) = 1\}.$$

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The theory of the lexicographic sum of Z

Example. Assume that $(G_\gamma)_{\gamma\in\Gamma}$ is a family of non-trivial archimedean ordered abelian groups, where $(\Gamma,<)$ is an ordered set. Consider the Hahn product

$$H=\{f\in\prod_{\gamma\in\Gamma}G_\gamma\ :\ f \text{ has well ordered support}\}.$$

Then the induced structure on the spine of H is $(\Gamma,<,C_\phi)_{\phi\in\mathfrak{L}_{oag}}$, where C_ϕ are unary predicates.

We focus on oag with spine $(\omega,<)$, and no colours.
Example: $\bigoplus_{i\in\omega^*}\mathbb{Z}$.

QE

Let $\mathfrak{L}$ be the language consisting of

- the main sort G with $+, -, 0, <, \equiv_m \ (m \in \mathbb{N})$;
- an auxiliary sort Γ with $<, 0, \infty, s : \Gamma \longrightarrow \Gamma$;
- $val^n : G \longrightarrow \Gamma \, (n \in \mathbb{N}, n \neq 1)$,
- an unary predicate $=^\bullet k_\bullet$ on G for each $k \in \mathbb{Z} \setminus \{0\}$,
- an unary predicate $\equiv^\bullet_m k_\bullet$ on G for each $m \geq 2$ *and* $k \in \{1, \dots, m-1\}$.

QE

Fact

Let G be an oag with spine ω and no colours. Then the theory of G has quantifier elimination in $\mathfrak{L}$, where

- $\Gamma = \omega \cup \{\infty\}$,
- $s(n) = n + 1$,
- *for every* $a \in G$, $val^n(a) := minsupp(a \mod nG)$ *if* $a \notin nG$, $val^n(a) := \infty$ *otherwise (or equivalently* $val^n(a)$ *is the index i of the largest convex subgroup* H_i *such that* $a \notin H_i + nG$*),*
- *for every* $a \in G$, $a =^{\bullet} k_{\bullet}$ *if* $a + H_i$ *is k times the minimal element of* G/H_i *for some* $i \in G$,
- *for every* $a \in G$, $a \equiv^{\bullet}_m k_{\bullet}$ *if* $a + H_i$ *is congruent modulo m to k times the minimal element of* G/H_i *for some convex subgroup* H_i.

Model completeness

Proposition (D.M.)

Let G be an oag with spine ω and no colours. Then the theory of G is model complete in the one sorted language $\mathfrak{L}$ consisting of

- $+, -, 0, <,$
- *for every* $n, m \in \mathbb{N}$ *a relation symbol* $|^{n,m}$,
- *for every* $n, m \in \mathbb{N}$ *a binary predicate* $\bar{s}^{n,m}$,
- *an unary predicate* $=^{\bullet} 1_{\bullet}$

where

- $x|^{n,m}y \iff val^n(x) \leq val^m(y)$,
- $\bar{s}^{n,m}(x, y) \iff val^m(y) = s(val^n(x))$,
- *for every* $a \in G$, $a =^{\bullet} 1_{\bullet}$ *if* $a + H_i$ *is the minimal element of* G/H_i *for some convex subgroup* H_i.

Model completeness with value group an oag with spine ω and no colours

Theorem (D.M.)

Let K be an Henselian valued field with the same properties and value group an oag G with spine ω and no colours. If $Th(k)$ is m.c. in $\mathfrak{L}_{rings}$, then $Th(K)$ is m.c. in $\mathfrak{L}_{rings}$ together with

- *for every $n, m \in \mathbb{N}$ a relation symbol $||^{n,m}$,*
- *for every $n, m \in \mathbb{N}$ a binary predicate $\$^{n,m}$,*
- *an unary predicate A,*

where

- *for every $x, y \in K$, $x||^{n,m}y \iff val^n(v(x)) \leq val^m(v(y))$,*
- *for every $x, y \in K$, $\$(x, y) \iff val^m(v(y)) = s(val^n(v(x)))$,*
- *$A^K = \{x \in K \mid v(x) =^{\bullet} 1_{\bullet}\}$.*

Example infinite many variables

This language gives model completeness for the following valued field. Consider the field of Hahn series over $\mathbb{Q}_p$ in infinitely many indeterminates

$$\mathcal{K} = \bigcup_{n \in \mathbb{N}} \mathbb{Q}_p((t_1^{\mathbb{Z}})) \dots ((t_n^{\mathbb{Z}})).$$

We can define, from the valuations val_n over $\mathbb{Q}_p((t_1^{\mathbb{Z}})) \dots ((t_n^{\mathbb{Z}}))$, a valuation val_∞ over $\mathcal{K}$ with values in $\bigoplus_{i<\omega*} \mathbb{Z}$, such that

$$O_{val_\infty} = \bigcup_{n \in \mathbb{N}} O_{val_n}.$$