Cyclic $\lambda\mu$-calculus

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Background

Curry-Howard

- Formulas as types
- Proofs as terms

Kolmogorov

- Formulas as problems
- Proofs as solutions

Game semantics

- Formulas as games
- Proofs as strategies

Cyclic proofs as programs

- Das: cyclic System **T**
- Das & Curzi: cyclic proofs for intuitionistic FPL
- Baelde, Doumane, Saurin & others: cyclic proofs for linear FPL
- Afshari, E. & Leigh: Herbrand schemes, cyclic proofs for classical logic with IP

Classical logic, “non-deterministic” approaches

- Herbrand’s theorem
- Alcolei, Clairambault, Hyland & Winskel: “true concurrency”
- Barbanera & Berardi: Symmetric lambda-calculus
- Bierman & Urban, strongly normalizing sequent calculus
- Herbrand schemes

Classical logic, “deterministic” approaches

Common feature is *confluence*.

- Girard’s “new constructive logic”
- Parigot’s $\lambda\mu$-calculus and variants
- Geuvers, Krebbers & McKinna: $\lambda\mu^{\mathsf{T}}$-calculus

Philosophical excursion

Parigot:

Usually when considering proofs as programs, one has only in mind some kind of intuitionistic proofs. There is an obvious reason for that restriction: only intuitionistic proofs are contructive, in the sense that from the proof of an existential statement, one can get a witness of this existential statement. But from the programming point of view, constructivity is only needed for Σ^0_1-statements, for which classical and intuitionistic provability coincide. This means that, classical proofs are also candidates for being programs.

Classical vs intuitionistic logic

- On the level of programs, no real difference - they let us *do the same things*
- Less distinctions between types, easier to construct well-typed programs
- But types less informative - it's a trade-off

Non-confluence and non-determinism

Hetzl:
... in a strikingly strong sense, the computational content of an arithmetical proof is *not a function*. As useful it is, from both a theoretical and a practical point of view, to extract a function from a proof, such extraction methods in general fall short of doing justice to the notion of computational content, as they cannot satisfy the unambiguity suggested by the term *content*.

- My view: classical sequent calculus is not "non-deterministic" but rather imprecise about its exact computational interpretation.
- We can make it precise by specifying reduction strategies; seems to be the view of Curien & Herbelin in Duality of Computation.
- Confluent systems have the precise interpretation built in.

Cyclic proofs for classical inductive types

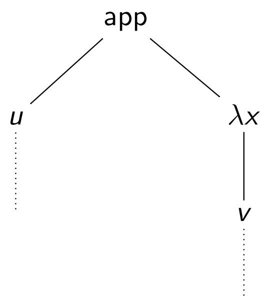
Cyclic proof terms with μ-operator

- Non-wellfounded terms and typing derivations
- Inductive types introduced by *constructors*
- Typing derivations involve dual *destructors*

Co-terms

$$u(\lambda x.v)$$

is really:



Simple type theory foundation

$$\frac{\Gamma \vdash u : A \qquad \Delta \vdash v : A \to B}{\Gamma, \Delta \vdash vu : B} \qquad \frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x.u : A \to B}$$

$$\frac{}{\Gamma, x : A \vdash x : A}$$

Inductive types

Natural numbers

$$\frac{u : N}{\kappa_N^1 u : N} \qquad \frac{}{\kappa_N^0 : N}$$

Lists

$$\frac{u : N \qquad v : L}{\kappa_L(u, v) : L} \qquad \frac{}{\kappa_L^0 : L}$$

Notation

- $\kappa_N^0 = 0$, $\kappa_N^1 n = sn$
- $\kappa_L^0 = \varepsilon$, $\kappa_L^1(u, v) = u \cdot v$

Wellfounded binary trees

$$\frac{}{\kappa^0_{BT} : BT} \qquad \frac{u : BT \qquad v : BT}{\kappa^1_{BT}(u, v) : BT}$$

Wellfounded ω-branching trees

$$\frac{}{\kappa^0_{OmT} : OmT} \qquad \frac{f : N \to OmT}{\kappa^1_{OmT} f : OmT}$$

Sums and products

$$\frac{u : A}{\kappa^0_{A+B} u : A + B} \qquad \frac{u : B}{\kappa^1_{A+B} u : A + B}$$

$$\frac{u : A \qquad v : B}{\kappa_{A \times B}(u, v) : A \times B}$$

Typing rules for inductive types

$$\frac{\Gamma \vdash u_1 : A_1 \qquad \cdots \qquad \Gamma \vdash u_n : A_n}{\Gamma \vdash \kappa^r_P(u_1, \ldots, u_n) : P}$$

$$\frac{\Gamma \vdash w : P \qquad \Delta, \vec{x}_1 : \vec{A}_1 \vdash u_1 : B \qquad \cdots \qquad \Delta, \vec{x}_k : \vec{A}_k \vdash u_k : B}{\Gamma, \Delta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_k(w, u_1, \ldots, u_k) : B}$$

Destructor for natural numbers

$$\frac{\Gamma \vdash u : N \qquad \Delta \vdash v : B \qquad \Delta, y : N \vdash w : B}{\Gamma, \Delta \vdash \delta_N y(u, v, w) : B}$$

Free and bound variables

$$\dfrac{\dfrac{x : N \vdash x : N}{x : N \vdash sx : N} \qquad \begin{array}{c}\vdots \\ \vdash u : A\end{array} \qquad \begin{array}{c}\vdots \\ x : N \vdash v(x) : A\end{array}}{x : N \vdash \delta_N x(sx, v(x)) : A}$$

Trace progression

$$\frac{z : P \vdash z : P \qquad \Gamma, \vec{x}_1 : \vec{A}_1 \vdash u_1 : B \qquad \cdots \qquad \Gamma, \vec{x}_k : \vec{A}_k \vdash u_k : B}{z : P, \Gamma \vdash \delta_P \vec{x}_1 \ldots \vec{x}_k(z, u_1, \ldots, u_k) : B}$$

Notation:

$$\frac{\Gamma, \vec{x}_1 : \vec{A}_1 \vdash u_1 : B \qquad \cdots \qquad \Gamma, \vec{x}_k : \vec{A}_k \vdash u_k : B}{z : P, \Gamma \vdash \delta_P \vec{x}_1 \ldots \vec{x}_k(z, u_1, \ldots, u_k) : B}$$

Example

$$
\dfrac{\dfrac{\dfrac{\kappa^0_E : E}{\kappa^0_+\kappa^0_E : E + O} \qquad \dfrac{\begin{matrix}\vdots \\ x : N \vdash u : E + O\end{matrix} \quad \dfrac{\dfrac{z_0 : E \vdash z_0 : E}{z_0 : E \vdash \kappa_O z_0 : O}}{z_0 : E \vdash \kappa^1_+\kappa_O z_0 : E + O} \quad \dfrac{\dfrac{z_1 : O \vdash z_1 : O}{z_1 : O \vdash \kappa^1_E z_1 : E}}{z_1 : O \vdash \kappa^0_+\kappa^1_E z_1 : E + O}}{x : N \vdash \delta_+ z_0 z_1(u, \kappa^1_+\kappa_O z_0, \kappa^0_+\kappa^1_E z_1) : E + O}}{x : N \vdash \delta_N x(x, \kappa^0_+\kappa^0_E, \delta_+ z_0 z_1(u, \kappa^1_+\kappa_O z_0, \kappa^0_+\kappa^1_E z_1)) : E + O}}{\vdash \lambda x.\delta_N x(x, \kappa^0_+\kappa^0_E, \delta_+ z_0 z_1(u, \kappa^1_+\kappa_O z_0, \kappa^0_+\kappa^1_E z_1)) : N \to E + O}
$$

Fixpoint equation

$$
u := \lambda x.\delta_N x(x, \kappa^0_+\kappa^0_E, \delta_+ z_0 z_1(u, \kappa^1_+\kappa_O z_0, \kappa^0_+\kappa^1_E z_1))
$$

Parigot-style classical logic

$$\frac{\Gamma, a : A \vdash u : \bot}{\Gamma \vdash \mu a.u : A} \qquad \frac{\Gamma \vdash u : A}{\Gamma, a : A \vdash [a]u : \bot}$$

Rules for substitutions

$$\frac{\Gamma, x : A \vdash u : B \qquad \Delta \vdash v : A}{\Gamma, \Delta \vdash u[x := v] : B} \qquad \frac{\Gamma, a : A \vdash u : \bot \qquad \Delta, x : A \vdash v : B}{\Gamma, \Delta, a : B \vdash u[a := \langle x \rangle v] : \bot}$$

Evaluating substitutions

$$x[x := u] \longrightarrow u$$

$$([a]v)[a := \langle x \rangle u] \longrightarrow [a](u[x := v])$$

Reducing λ

$$(\lambda x.u)v \longrightarrow u[x := v]$$

Reducing μ

$$(\mu a.u)v \longrightarrow \mu a.u[a := \langle z \rangle (zv)]$$

$\delta\kappa$-rule

$$\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathsf{Pr}(P)|}(\kappa_P^r(\vec{u}), v_1, \ldots, v_{|\mathsf{Pr}(P)|}) \longrightarrow v_r[\vec{x}_r := \vec{u}]$$

$\delta\kappa$-rule

$$\dfrac{\dfrac{\{\Gamma \vdash v_j : A^r_j\}_{j \leq \mathbf{ar}(r)}}{\Gamma \vdash \kappa^r_P(\vec{v}_r) : P} \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(\kappa^r_P(\vec{v}_r), u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}$$

$$\Downarrow$$

$$\dfrac{\{\Gamma \vdash v_j : A^r_j\}_{j \leq \mathbf{ar}(r)} \qquad \Delta, \vec{x}_r : \vec{A}_r \vdash u_r : B}{\Gamma, \Delta \vdash u_r[\vec{x}_r := \vec{v}_r] : B}$$

$\delta\mu$-rule

$$\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(\mu a.w, v_1, \ldots, v_{|\mathrm{Pr}(P)|})$$
$$\longrightarrow \mu a.w[a := \langle z \rangle \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(z, v_1, \ldots, v_{|\mathrm{Pr}(P)|})]$$

$\delta\mu$-rule

$$\dfrac{\dfrac{\Gamma, a : P \vdash w : \bot}{\Gamma \vdash \mu a.w : P} \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \le |\mathsf{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_i \ldots \vec{x}_{|\mathsf{Pr}(P)|}(\mu a.w, u_1, \ldots, u_{|\mathsf{Pr}(P)|}) : B}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma, a : P \vdash w : \bot \qquad \dfrac{\{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \le |\mathsf{Pr}(P)|}}{\Delta, z : P \vdash \delta_P(z, u_1, \ldots, u_{|\mathsf{Pr}(P)|}) : B}}{\Gamma, \Delta, a : B \vdash w[a := \langle z \rangle \delta_P(z, u_1, \ldots, u_{|\mathsf{Pr}(P)|})] : \bot}}{\Gamma, \Delta \vdash \mu a.w[a := \langle z \rangle \delta_P(z, u_1, \ldots, u_{|\mathsf{Pr}(P)|})] : B}$$

$\delta\delta$-rule

$$\delta_Q \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(Q)|}(\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, v_1, \ldots, v_{|\mathrm{Pr}(P)|}), \vec{w})$$
$$\longrightarrow \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, \delta_Q \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(Q)|}(v_1, \vec{w}), \ldots, \delta_Q \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(Q)|}(v_{|\mathrm{Pr}(P)|}, \vec{w}))$$

$\delta\delta$-rule

$$\frac{\dfrac{\Gamma \vdash u : P \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : Q\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : Q} \qquad \{\Theta, \vec{y}_i : \vec{B}_i \vdash v_i : C\}_{i \leq |\mathrm{Pr}(Q)|}}{\Gamma, \Delta, \Theta \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}), \vec{v}) : C}$$

$$\Downarrow$$

$$\frac{\Gamma \vdash u : P \qquad \{\Delta, \Theta, \vec{x}_i : \vec{A}_i \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_i, \vec{v}) : C\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta, \Theta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_2, \vec{v}), \ldots, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_{|\mathrm{Pr}(P)|}, \vec{v})) : C}$$

Deriving Parigot's μ-reduction

$$\dfrac{\dfrac{\vdots}{\Delta \vdash w : A} \qquad \dfrac{\dfrac{\vdots}{\Gamma, a : A \to B \vdash u([a]v) : \bot}}{\Gamma \vdash \mu a.u([a]v) : A \to B}}{\Gamma, \Delta \vdash (\mu a.u([a]v))w : B}$$

Reduction:

$$\begin{aligned}
(\mu a.u([a]v))w &\longrightarrow \mu a.u([a]v)[a := \langle z\rangle(zw)] \\
&\longrightarrow^* \mu a.u(([a]v)[a := \langle z\rangle(zw)]) \\
&\longrightarrow \mu a.u([a]((zw)[z := w])) \\
&\longrightarrow \mu a.u([a]vw)
\end{aligned}$$

Why extra reductions for μ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ x : B \vdash u : N\end{array}}{x : B \vdash su : N}}{x : B, a : N \vdash [a]su : \bot}}{a : N \vdash \lambda x.[a]su : \neg B}}{b : \neg B, a : N \vdash [b]\lambda x.[a]su : \bot}}{b : \neg B \vdash \mu a.[b]\lambda x.[a]su : N} \qquad \begin{array}{c}\vdots \\ \vdash v : C\end{array} \qquad \begin{array}{c}\vdots \\ z : N \vdash w : C\end{array}}{b : \neg B \vdash \delta_N z(\mu a.[b]\lambda x.[a]su, v, w) : C}
$$

Reduction sequence

$$
\begin{aligned}
\delta_N z(\mu a.[b]\lambda x.[a]su, v, w) &\longrightarrow \mu a.([b]\lambda x.[a]su)[a := \langle y\rangle \delta_N z(y, v, w)] \\
&\longrightarrow^* \mu a.([b]\lambda x.([a]su)[a := \langle y\rangle \delta_N z(y, v, w)]) \\
&\longrightarrow \mu a.([b]\lambda x.[a](\delta_N z(y, v, w))[y := su]) \\
&\longrightarrow^* \mu a.([b]\lambda x.[a](\delta_N z(su, v, w)))
\end{aligned}
$$

New typing derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ x : B \vdash u : N\end{array}}{x : B \vdash su : N} \qquad \begin{array}{c}\vdots\\ \vdash v : C\end{array} \qquad \begin{array}{c}\vdots\\ z : N \vdash w : C\end{array}}{x : B \vdash \delta_N z(su, v, w) : C}}{x : B, a : C \vdash [a]\delta_N z(su, v, w) : \bot}}{a : C \vdash \lambda x.[a]\delta_N z(su, v, w) : \neg B}}{a : C, b : \neg B \vdash [b]\lambda x.[a]\delta_N z(su, v, w) : \bot}}{b : \neg B \vdash \mu a.[b]\lambda x.[a]\delta_N z(su, v, w) : C}
$$

Continuing the reduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots \\ x : B \vdash u : N & & z : N \vdash w : C\end{matrix}}{x : B \vdash w[z := u] : C}}{x : B, a : C \vdash [a]\delta_N z(su, v, w) : \bot}}{a : C \vdash \lambda x.[a]\delta_N z(su, v, w) : \neg B}}{a : C, b : \neg B \vdash [b]\lambda x.[a]\delta_N z(su, v, w) : \bot}}{b : \neg B \vdash \mu a.[b]\lambda x.[a]\delta_N z(su, v, w) : C}$$

Avoiding obvious non-confluence

- Parigot: $u(\mu a.v) \longrightarrow \mu a.v[a := \langle x \rangle u(x)]$
- Unrestricted use leads to non-confluence: $(\lambda x.u)(\mu a.v)$
- Duality of Computation: call-by-name vs. call-by-value
- Here, we call $\mu a.v$ by name in argument position *unless* evalution is needed to evaluate a destructor

Subject reduction in the limit

$\delta\kappa$-rule

$$\dfrac{\dfrac{\{\Gamma \vdash v_j : A^r_j\}_{j \leq \mathbf{ar}(r)}}{\Gamma \vdash \kappa^r_P(\vec{v}_r) : P} \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \leq |\mathsf{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathsf{Pr}(P)|}(\kappa^r_P(\vec{v}_r), u_1, \ldots, u_{|\mathsf{Pr}(P)|}) : B}$$

$$\Downarrow$$

$$\dfrac{\{\Gamma \vdash v_j : A^r_j\}_{j \leq \mathbf{ar}(r)} \qquad \Delta, \vec{x}_r : \vec{A}_r \vdash u_r : B}{\Gamma, \Delta \vdash u_r[\vec{x}_r := \vec{v}_r] : B}$$

$\delta\mu$-rule

$$\dfrac{\dfrac{\Gamma, a : P \vdash w : \bot}{\Gamma \vdash \mu a.w : P} \qquad \{\Delta, \vec{x_i} : \vec{A_i} \vdash u_i : B\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x_i} \ldots \vec{x}_{|\mathrm{Pr}(P)|}(\mu a.w, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma, a : P \vdash w : \bot \qquad \dfrac{\{\Delta, \vec{x_i} : \vec{A_i} \vdash u_i : B\}_{i \leq |\mathrm{Pr}(P)|}}{\Delta, z : P \vdash \delta_P(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}}{\Gamma, \Delta, a : B \vdash w[a := \langle z \rangle \delta_P(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|})] : \bot}}{\Gamma, \Delta \vdash \mu a.w[a := \langle z \rangle \delta_P(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|})] : B}$$

$\delta\delta$-rule

$$\dfrac{\dfrac{\Gamma \vdash u : P \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : Q\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : Q} \qquad \{\Theta, \vec{y}_i : \vec{B}_i \vdash v_i : C\}_{i \leq |\mathrm{Pr}(Q)|}}{\Gamma, \Delta, \Theta \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}), \vec{v}) : C}$$

$$\Downarrow$$

$$\dfrac{\Gamma \vdash u : P \qquad \{\Delta, \Theta, \vec{x}_i : \vec{A}_i \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_i, \vec{v}) : C\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta, \Theta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_2, \vec{v}), \ldots, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_{|\mathrm{Pr}(P)|}, \vec{v})) : C}$$

Special case

$$\cfrac{\cfrac{\{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : Q\}_{i \leq |\mathrm{Pr}(P)|}}{\Delta, z : P \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : Q} \qquad \{\Theta, \vec{y}_i : \vec{B}_i \vdash v_i : C\}_{i \leq |\mathrm{Pr}(Q)|}}{\Delta, z : P, \Theta \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(\delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|}), \vec{v}) : C}$$

$$\Downarrow$$

$$\cfrac{\{\Delta, \vec{x}_i : \vec{A}_i, \Theta, \vdash \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_i, \vec{v}) : C\}_{i \leq |\mathrm{Pr}(P)|}}{\Delta, z : P, \Theta \vdash \delta_P \vec{x}_1 \ldots \vec{x}_{|\mathrm{Pr}(P)|}(z, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_1, \vec{v}), \ldots, \delta_Q \vec{y}_1 \ldots \vec{y}_{|\mathrm{Pr}(Q)|}(u_{|\mathrm{Pr}(P)|}, \vec{v})) : C}$$

Substitutions for assumption variables

$$\dfrac{\Gamma \vdash u : P \qquad \dfrac{\{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \leq |\mathrm{Pr}(P)|}}{\Delta, z : P \vdash \delta_P \vec{x}_i \ldots \vec{x}_{|\mathrm{Pr}(P)|}(z, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}}{\Gamma, \Delta \vdash \delta_P \vec{x}_i \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}$$

$$\Downarrow$$

$$\dfrac{\Gamma \vdash u : P \qquad \{\Delta, \vec{x}_i : \vec{A}_i \vdash u_i : B\}_{i \leq |\mathrm{Pr}(P)|}}{\Gamma, \Delta \vdash \delta_P \vec{x}_i \ldots \vec{x}_{|\mathrm{Pr}(P)|}(u, u_1, \ldots, u_{|\mathrm{Pr}(P)|}) : B}$$

Substitutions for counter-assumption variables

$$\dfrac{\dfrac{\Gamma \vdash u : A}{\Gamma, a : A \vdash [a]u : \bot} \qquad \Delta, z : A \vdash v : B}{\Gamma, \Delta, a : B \vdash : ([a]u)[a := \langle z \rangle v] : \bot}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma \vdash u : A \qquad \Delta, z : A \vdash v : B}{\Gamma, \Delta \vdash v[z := u] : B}}{\Gamma, \Delta, a : B \vdash [a](v[z := u]) : \bot}$$

Test case

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ x : P \vdash u : P\end{array}}{x : P \vdash \kappa_P u : P}}{x : P \vdash \delta_P x(x, \kappa_P u) : P} \qquad \dfrac{\begin{array}{c}\vdots\\ x : P \vdash v : Q\end{array}}{x : P \vdash \kappa_Q v : Q}}{x : P \vdash \delta_P x(\delta_P x(x, \kappa_P u), \kappa_Q v) : Q}
$$

Fixpoint equations

$$
u = \delta_P x(x, \kappa_P u)
$$

$$
v = \delta_P x(x, \kappa_Q v)
$$

Reduction

$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\vdots \\ {\color{red}x} : P \vdash u : P\end{array}}{{\color{red}x} : P \vdash \kappa_P u : P}
}{{\color{red}x} : P \vdash \delta_P x(x, \kappa_P u) : P}
\qquad
\dfrac{\begin{array}{c}\vdots \\ {\color{blue}x} : P \vdash v : Q\end{array}}{{\color{blue}x} : P \vdash \kappa_Q v : Q}
}{{\color{red}x} : P \vdash \delta_P x(\delta x(x, \kappa_P u), \kappa_Q v) : Q}
$$

Reduction

$$\dfrac{\dfrac{\dfrac{\vdots}{x : P \vdash u : P}}{x : P \vdash \kappa_P u : P} \qquad \dfrac{\dfrac{\vdots}{x : P \vdash v : Q}}{x : P \vdash \kappa_Q v : Q}}{\dfrac{x : P \vdash \delta_P(\kappa_P u, \kappa_Q v) : Q}{x : P \vdash \delta_P x(x, \delta_P(\kappa_P u, \kappa_Q v)) : Q}}$$

Reduction

$$
\dfrac{\dfrac{\vdots}{x : P \vdash u : P} \qquad \dfrac{\dfrac{\vdots}{x : P \vdash v : Q}}{x : P \vdash \kappa_Q v : Q}}{\dfrac{x : P \vdash \kappa_Q v[x := u] : Q}{x : P \vdash \delta_P x(x, \kappa_Q v[x := u]) : Q}}
$$

Reduction

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc} \vdots & \vdots \\ x : P \vdash u : P & x : P \vdash v : Q \end{array}}{x : P \vdash v[x := u] : Q}}{x : P \vdash \kappa_Q v[x := u] : Q}}{x : P \vdash \delta_P x(x, \kappa_Q v[x := u]) : Q}$$

Reduction

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ x : P \vdash u : P\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ x : P \vdash \kappa_Q v : Q\end{array}}{x : P \vdash \delta_P(x, \kappa_Q v) : Q}}{x : P \vdash v[x := u] : Q}}{x : P \vdash \kappa_Q v[x := u] : Q}}{x : P \vdash \delta_P x(x, \kappa_Q v[x := u]) : Q}
$$

Reduction

$$
\dfrac{
  \dfrac{\dfrac{
    \begin{array}{c}\vdots\\ x : P \vdash u : P\end{array}
    \qquad
    \dfrac{
      \begin{array}{c}\vdots\\ x : P \vdash u : P\end{array}
      \qquad
      \begin{array}{c}\vdots\\ x : P \vdash \kappa_Q v : Q\end{array}
    }{x : P \vdash \kappa_Q v[x := u] : Q}
  }{x : P \vdash \delta_P(u, \kappa_Q v[x := u]) : Q}}{x : P \vdash \kappa_Q \delta_P(u, \kappa_Q v[x := u]) : Q}
}{x : P \vdash \delta_P x(x, \kappa_Q \delta_P(u, \kappa_Q v[x := u])) : Q}
$$

Finitary types, confluence and strong normalization

Finitary types

Built up using only inductive predicates, no function types.
Examples: natural numbers, lists of natural numbers...

Proposition

If t is in normal form and $\vdash t : A$ where A is a finitary type then t is a finite wellfounded term.

Question

Can we prove confluence and normalization for terms of finitary type?

- Left trace condition as a "termination" condition as in cyclic **T**
- Need the right notion of *reducibility candidate*
- *Strong* normalization probably requires some form of "lazy evaluation" in order not to get stuck in non-wellfounded part

Failure of naïve strong normalization

$$
\dfrac{\dfrac{N}{N} \qquad \dfrac{\dfrac{\dfrac{E}{E+O} \qquad \dfrac{\begin{matrix}\vdots \\ N \vdash E+O\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{E \vdash E}{E \vdash O}}{E \vdash E+O} \qquad \dfrac{\dfrac{O \vdash O}{O \vdash E}}{O \vdash E+O}}{E+O \vdash E+O}}{\vdash E+O \to E+O}}{N \vdash E+O}}{N \vdash E+O}}{\vdash N \to E+O}}{E+O}
$$

Unfolding

$$
\dfrac{\dfrac{N}{N} \qquad \dfrac{\dfrac{\dfrac{E}{E+O} \qquad \dfrac{\dfrac{\dfrac{E}{E+O} \qquad \dfrac{\begin{matrix}\vdots \\ N \vdash E+O\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E \vdash E}{E \vdash O}}{E \vdash E+O} \qquad \dfrac{\dfrac{O \vdash O}{O \vdash E}}{O \vdash E+O}}{E+O \vdash E+O}}{\vdash E+O \to E+O}}{N \vdash E+O}}{N \vdash E+O} \qquad \dfrac{\dfrac{\dfrac{\dfrac{E \vdash E}{E \vdash O}}{E \vdash E+O} \qquad \dfrac{\dfrac{O \vdash O}{O \vdash E}}{O \vdash E+O}}{E+O \vdash E+O}}{\vdash E+O \to E+O}}{N \vdash E+O}}{N \vdash E+O}}{\vdash N \to E+O}}{E+O}
$$

$\beta + \delta\delta$

$$\dfrac{\dfrac{N}{N}\qquad \dfrac{\dfrac{\dfrac{E}{E+O}\qquad \dfrac{\dfrac{\dfrac{E}{E+O}\quad \dfrac{\dfrac{\dfrac{E\vdash E}{E\vdash O}}{E\vdash E+O}\quad \dfrac{\dfrac{O\vdash O}{O\vdash E}}{O\vdash E+O}}{E+O\vdash E+O}}{\vdash E+O}\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ N\vdash E+O\end{matrix}\quad \dfrac{\dfrac{\dfrac{\dfrac{E\vdash E}{E\vdash O}}{E\vdash E+O}\quad \dfrac{\dfrac{O\vdash O}{O\vdash E}}{O\vdash E+O}}{E+O\vdash E+O}}{\vdash E+O\to E+O}}{N\vdash E+O}\quad \dfrac{\dfrac{\dfrac{E\vdash E}{E\vdash O}}{E\vdash E+O}\quad \dfrac{\dfrac{O\vdash O}{O\vdash E}}{O\vdash E+O}}{E+O\vdash E+O}}{N\vdash E+O}}{N\vdash E+O}}{N\vdash E+O}}{\vdash N\to E+O}}{E+O}$$

Concluding remarks

- Still many basic things to work out
- CPS-translation?
  - Results by Das, Pous & Curzi on intuitionistic μ-calculus
- Computational expressivity
- Ordinal variables?
  - Seem to be related to "bouncing threads"
  - Could they simplify normalization?
- Categorical models?
  - Ong has a fibrational semantics of $\lambda\mu$-calculus
  - Santocanale: μ-bicomplete categories
  - Do they mix?
- Game models?