Rigidity and dynamics

I. Farah (with B. De Bondt, A. Vignati, and W. Brian)

LLAMAS, Amsterdam, October 16, 2024

III. First-order theories of $\mathfrak{B}_n = (\mathcal{P}(n), \sigma_n)$

Let σ_n denote the automorphism of the Boolean algebra $\mathcal{P}(n)$ obtained by cycling its atoms.

Problem

Describe sequences $(n(j))$ such that the first-order theories of $\langle \mathcal{P}(n(j)), \sigma_{n(j)} \rangle$ converge.

The *monadic second-order logic* is the extension of the first-order logic in which quantification over subsets of the domain is allowed.

The Very Same Problem

Describe sequences $(n(j))$ such that the monadic second-order theories of (directed) $n(j)$-cycles converge.

I. Definitions. Reduced product $\mathcal{M} = \prod_i M_i/\mathcal{I}$

Suppose that M_i, for $i \in \mathbb{N}$, are structures in the same language $\mathcal{L}$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$. For (a_i) and (b_i) in $\prod_i M_i$ let

$$(a_i) =^{\mathcal{I}} (b_i) \Leftrightarrow (\forall^{\mathcal{I}} i) a_i = b_i \qquad (\Leftrightarrow \{i \mid a_i \neq b_i\} \in \mathcal{I}).$$

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For every $\mathcal{L}$-relation symbol $R(x, y)$, set

$$R^{\mathcal{M}}([(a_i)], [(b_i)]) \Leftrightarrow (\forall^{\mathcal{I}} i) R^{M_i}(a_i, b_i).$$

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(Similarly for n-ary relation symbols.)
In this talk I will consider only the *Frechét ideal*

$$\text{Fin} = \{A \subseteq \mathbb{N} \mid |A| < \aleph_0\}.$$

Basic question: Rigidity od reduced products

Suppose that M_i, N_i, for $i \in \mathbb{N}$, are countable structures of the same countable language.

1. Can we describe all automorphisms of $\prod_i M_i / \mathrm{Fin}$?
2. When is $\prod_i M_i / \mathrm{Fin} \cong \prod_i N_i / \mathrm{Fin}$?

A tale of two examples

Example

The Boolean group, $\prod_i (\mathbb{Z}/2\mathbb{Z})/\operatorname{Fin}$.

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1. It has $2^{2^{\aleph_0}}$ automorphisms.
2. For all ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$,

$$\prod_i M_i/\mathcal{I} \cong \prod_i M_i/\mathcal{J} \quad \Leftrightarrow \quad |\mathcal{P}(\mathbb{N})/\mathcal{I}| = |\mathcal{P}(\mathbb{N})/\mathcal{J}|.$$

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The Boolean algebra $\prod_i \{0,1\}/\operatorname{Fin}$

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Example

The Boolean algebra $\prod_i \{0,1\}/\operatorname{Fin} \cong \mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

A case study: Can we describe all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$?

An *almost permutation* of $\mathbb{N}$ is a bijection between cofinite subsets of $\mathbb{N}$.

Every almost permutation γ defines an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ by

$$\alpha_\gamma([A]) = [\gamma[A]].$$

Automorphisms of the form α_γ are called *trivial*.

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Lemma

The index map

$$\operatorname{index}(\alpha_\gamma) = |\mathbb{N} \setminus \operatorname{dom}(\gamma)| - |\mathbb{N} \setminus \operatorname{range}(\gamma)|$$

defines a homomorphism from the group of trivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ onto $\mathbb{Z}$.

CH vs. some forcing extension

Theorem (W. Rudin, 1956)

The Continuum Hypothesis, CH, implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *has* $2^{2^{\aleph_0}}$ *nontrivial automorphisms.*

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Theorem (Shelah, 1979)

In some forcing extension of the universe all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ *are trivial.*

Forcing axioms: An alternative to CH

Assume $\mathbb{K}$ is a class of compact Hausdorff spaces and consider a strengthening of the Baire Category Theorem.

FA($\mathbb{K}$) If $\Omega \in \mathbb{K}$, then an intersection of $\aleph_1$ dense open subsets of Ω is dense.

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Theorem (Foreman–Magidor–Shelah, 1988)

There is a maximal class $\mathbb{K}$ for which FA($\mathbb{K}$) (known as Martin's Maximum, MM) is relatively consistent with ZFC, modulo large cardinal axioms.

Weaker forcing axioms include Proper Forcing Axiom (PFA), Martin's Axiom (MA). Open Colouring Axiom $\mathrm{OCA_T}$ is a consequence of PFA.

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Theorem

Forcing axioms imply that all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are trivial: Shelah–Steprāns (1988, PFA), Veličković (1993, $\mathrm{OCA_T}$+MA), De Bondt–F.–Vignati, 2024 ($\mathrm{OCA_T}$).

The real result behind Rudin's theorem

Lemma

Assume CH.

1. $\prod_i M_i/\operatorname{Fin}$ *has* $2^{2^{\aleph_0}}$ *automorphisms (most of them nontrivial, for any reasonable definition of trivial automorphisms).*
2. $\prod_i M_i/\operatorname{Fin} \cong \prod_i N_i/\operatorname{Fin} \Leftrightarrow \prod_i M_i/\operatorname{Fin} \equiv \prod_i N_i/\operatorname{Fin}$
($\equiv$ is elementary equivalence).

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Proof.

$\prod_i M_i/\operatorname{Fin}$ is $\aleph_1$-saturated (Jónsson–Olin, 1968), and by CH, $\aleph_1 = 2^{\aleph_0}$. □

Basic question revisited: Rigidity od reduced products

Suppose that M_i, N_i, for $i \in \mathbb{N}$, are countable structures of the same countable language.

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Basic question revisited: Rigidity od reduced products

Suppose that M_i, N_i, for $i \in \mathbb{N}$, are countable structures of the same countable language. Assume CH.

1. Can we describe all automorphisms of $\prod_i M_i / \operatorname{Fin}$? No; this is a saturated model.
2. When is $\prod_i M_i / \operatorname{Fin} \cong \prod_i N_i / \operatorname{Fin}$?

By $\aleph_1$-saturation, CH reduces 2. to

2' When is $\prod_i M_i / \operatorname{Fin} \equiv \prod_i N_i / \operatorname{Fin}$?

The first-order theory of $\mathcal{M} = \prod_i M_i/\mathcal{I}$

Theorem (Fundamental Theorem on Ultraproducts. Łoś's, 1956)

If $\mathcal{I}$ is a maximal ideal, then every sentence φ satisfies $\mathcal{M} \models \varphi \Leftrightarrow (\forall^{\mathcal{I}} i) M_i \models \varphi$.

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Theorem (Feferman–Vaught, 1967)

For every $\mathcal{I}$, the $\mathrm{Th}(\mathcal{M})$ is computable from $\langle \mathrm{Th}(M_i) \mid i \in \mathcal{I} \rangle$ and $\mathcal{P}(\mathbb{N})/\mathcal{I}$.

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Theorem (E.A. Palyutin, 1980)

There is a set of formulas (called h-formulas) such that $\mathcal{M} \models \varphi \Leftrightarrow (\forall^{\mathcal{I}} i) M_i \models \varphi$.
If $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is an atomless Boolean algebra, then $\mathrm{Th}(\mathcal{M})$ implies that every formula is a Boolean combination of h-formulas.

The dividing line for non-rigidity (what's so special about $\prod_i(\mathbb{Z}/2\mathbb{Z})/\operatorname{Fin}$?)

Theorem (De Bondt–F.–Vignati, 2024)

1. *If $\prod_i M_i/\operatorname{Fin}$ has stable theory, then it is $2^{\aleph_0}$-saturated. In particular, it has $2^{2^{\aleph_0}}$ (nontrivial) automorphisms, provably in ZFC.*
2. *If $\prod_i M_i/\operatorname{Fin}$ does not have stable theory, then it is not $\aleph_2$-saturated.*

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Theorem (De Bondt–F.–Vignati, 2023)

$\mathrm{OCA_T} + \mathrm{MA}$ *imply that all isomorphism between reduced products over* Fin *of models of* *certain theories* *are* *trivial.*

Corollary (De Bondt–F.–Vignati, 2023)

Assume $\mathrm{OCA_T} + \mathrm{MA}$ *and that* $2 \le |M_i|, |N_i| \le \aleph_0$ *for all* i*.*

1. *If* M_i, N_i *are fields, then* $\prod_i M_i/\mathrm{Fin} \cong \prod_i N_i/\mathrm{Fin}$ *if and only if there is an almost permutation* γ *such that* $M_i \cong N_{\gamma(i)}$ *for all* i*.*
2. *If* M_i, N_i *are linear orderings then* $\prod_i M_i/\mathrm{Fin} \cong \prod_i N_i/\mathrm{Fin}$ *if and only if there is an almost permutation* γ *such that* $M_i \cong N_{\gamma(i)}$ *for all* i*.*
3. *If* M_i, N_i *are sufficiently random graphs then* $\prod_i M_i/\mathrm{Fin} \cong \prod_i N_i/\mathrm{Fin}$ *if and only if there is an almost permutation* γ *such that* $M_i \cong N_{\gamma(i)}$ *for all* i*.*

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Question

Is there a model-theoretic characterization of theories T *such that all automorphisms between reduced products of models ot* T *respect coordinates?*

II. Dynamics

Let σ denote the shift on $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$:

$$\sigma([A]) = [\{n+1 \mid n \in A\}].$$

Then $\operatorname{index}(\sigma) = -1$, $\operatorname{index}(\sigma^{-1}) = 1$.
For an automorphism α of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, write $\mathfrak{A}_\alpha$ for $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \alpha)$.

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Corollary

For all m, n *in* $\mathbb{Z}$, *we have*

1. *CH implies* $\mathfrak{A}_{\sigma^m} \cong \mathfrak{A}_{\sigma^n} \Leftrightarrow |m| = |n|$.
2. $\mathrm{OCA_T}$ *implies* $\mathfrak{A}_{\sigma^m} \cong \mathfrak{A}_{\sigma^n} \Leftrightarrow m = n$.

From this point on, all results are due to Brian–F.

Let $\bar{n} = (n(j))_j$ be a sequence in $\mathbb{N}$ such that $\lim_j n(j) = \infty$. Let $\gamma_{\bar{n}}$ be a permutation of $\mathbb{N}$ such that

1. $\mathbb{N} = \bigsqcup_j J(j)$ is a partition into intervals,
2. $|J(j)| = n(j)$ for all j,
3. $\gamma_{\bar{n}} \restriction J(j)$ is an $n(j)$-cycle, denoted σ_j.

This determines a trivial automorphism $\alpha_{\bar{n}}$ of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Write $\mathfrak{A}_{\bar{n}} = \mathfrak{A}_{\alpha_{\bar{n}}}$.

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Proposition

$(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \alpha_{\bar{n}}) \cong \prod_j \mathfrak{B}_{n(j)}/\operatorname{Fin}$.

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Proposition

$(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \alpha_{\bar{n}}) \cong \prod_j \mathfrak{B}_{n(j)}/\operatorname{Fin}$.
In particular, $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \alpha_{\bar{n}})$ *is countably saturated and CH implies that* $\mathfrak{A}_{\bar{m}} \cong \mathfrak{A}_{\bar{n}} \Leftrightarrow \mathfrak{A}_{\bar{m}} \equiv \mathfrak{A}_{\bar{n}}$.

Ghasemi's trick

Proposition (Ghasemi, 2016)

Every sequence (M_i) *of first-order structures of the same countable language has a subsequence* $(M_{k(i)})$ *such that every further subsequence* $(M_{l(i)})$ *satisfies* $\prod_i M_{k(i)}/\,\mathrm{Fin} \equiv \prod_i M_{l(i)}/\,\mathrm{Fin}$.

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Proof.

Choose $k(i)$ so that $\lim_{i\to\infty} \operatorname{Th}(M_{k(i)})$ exists, apply Feferman–Vaught. □

Corollary

There are $\bar{m}$, $\bar{n}$ such that $\mathfrak{A}_{\bar{m}} \cong \mathfrak{A}_{\bar{n}}$ is independent from ZFC.

This implies that the isomorphism of uniform Roe coronas of some uniformly locally finite metric spaces of asymptotic dimension 1 is independent from ZFC, but never mind that.

First-order theories of $\mathfrak{A}_{\bar{n}}$ and of $\mathfrak{B}_n = (\mathcal{P}(n), \sigma_n)$

The sequence $\mathrm{Th}(\mathfrak{B}_n)$, for $n \in \mathbb{N}$, is recursive. $\mathrm{Th}(\mathfrak{A}_{\bar{n}})$ is decidable by Feferman–Vaught but we don't know what it is.

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Lemma

For every $d \geq 2$, the assertion $(\forall^{\mathrm{Fin}} i) d | n_i$ is is first-order expressible in $\mathfrak{A}_{\bar{n}}$.

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Question

Is $\mathfrak{A}_{(2^{2n})} \equiv \mathfrak{A}_{(2^{2n+1})}$?

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Answer

No.

III (again). First-order theories of $\mathfrak{A}_{\bar{n}}$ and of $\mathfrak{B}_n = (\mathcal{P}(n), \sigma_n)$

The following is the best that we can do.

Lemma

For all $0 \leq r < d$*, the assertion* $(\forall^{\mathrm{Fin}} i) n_i \equiv r \pmod d$ *is first-order expressible in* $\mathfrak{A}_{\bar{n}}$*.*

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Problem

Describe sequences (n_j) such that the first-order theories of $\langle \mathcal{P}(n_j), r_{n_j} \rangle$ converge.

Equivalently, describe sequences (n_i) such that the monadic second-order theories of (directed) n_i-cycles converge.

For more on corona rigidity see

I. Farah, S. Ghasemi, A. Vaccaro, A. Vignati. *Corona rigidity*. arXiv:2201.11618 (2022).