Graded Monads & Behavioural Equivalence Games

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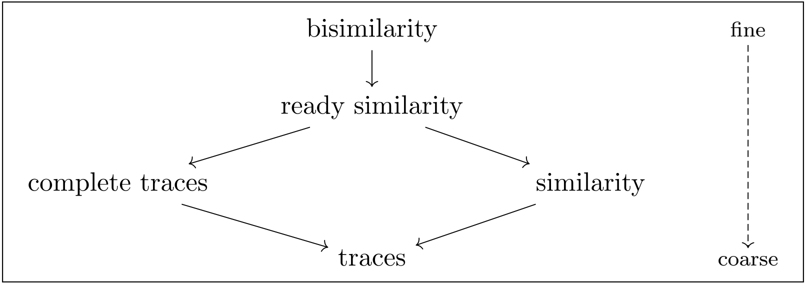
Where are we?



Linear-time–Branching-time spectrum for LTS (Van Glabbeek, 1990)

`Graded semantics`: framework for spectra of behavioural semantics

coalgebra [system-type] + graded monads [granularity]

Overview

I Aspects of graded monads

II Graded coalgebraic semantics

III Coalgebraic determinization under graded semantics

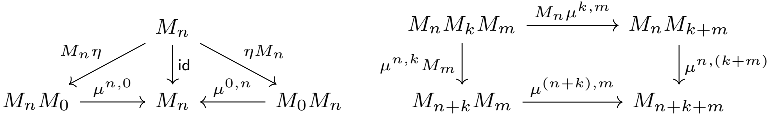
IV Game characterizations of graded semantics, for free

- Graded monads -

Graded monads

Graded monad $\mathbb{M} = (M, \eta, \mu)$ on Set:

$$(M_n : \mathsf{Set} \to \mathsf{Set})_{n \in \mathbb{N}} \qquad \eta : \mathsf{Id} \to M_0 \qquad (\mu^{n,k} : M_n M_k \to M_{n+k})_{n,k \in \mathbb{N}}$$



Abstractly: lax monoidal functor $(\mathbb{N}, +, 0) \to ([\mathsf{Set}, \mathsf{Set}], \circ, \mathsf{Id})$

Examples

Functor iteration

Given a functor $G\colon \mathsf{Set} \to \mathsf{Set}$, define $\mathbb{M}_G$ by

$$M_n := G^n \qquad \eta := \mathsf{Id} \xrightarrow{\mathsf{id}} G^0 \qquad \mu^{n,k} := G^n G^k \xrightarrow{\mathsf{id}} G^{n+k}$$

Kleisli distributive laws

Each distributive law $\lambda\colon FT \to TF$ with

$$(T, \eta, \mu) \text{ a monad} \qquad F\colon \mathsf{Set} \to \mathsf{Set} \text{ a functor}$$

yields a graded monad with $M_n := TF^n$, unit η, and multiplication

$$\mu^{n,k} := TF^n TF^k \xrightarrow{T\lambda^n F^k} TTF^n F^k \xrightarrow{\mu F^{n+k}} TF^{n+k} \qquad (\lambda^n\colon F^n T \to TF^n)$$

e.g. for each set $\mathcal{A}$, we have a graded monad with $M_n = \mathscr{P}_f(\mathcal{A}^n \times -)$

Graded algebras

- Graded monads admit a notion of *graded algebra* [FKM16, MPS15]

  ...generalizing the EM category of vanilla monads

- We consider refinements that interpret terms of u.d. at most n:

$$(A_k)_{k \leq n} \quad \text{(carrier)} \qquad\qquad (M_m A_k \xrightarrow{a^{m,k}} A_{m+k})_{m+k \leq n} \quad \text{(structure)}$$

- The category $\mathsf{Alg}_n(\mathbb{M})$ of M_n-algebras and homomorphisms has a forgetful functor

$$U\colon \mathsf{Alg}_n(\mathbb{M}) \to \mathsf{Set}, \qquad \mathbf{A} \mapsto A_0$$

Free M_n-algebra [MPS15]

The free M_n-algebra on a set X w.r.t. U has

- carrier: $(M_k X)_{k \leq n}$
- structure: $\mu^{\ell,k}\colon M_\ell M_k X \to M_{\ell+k} X$

The universal morphism is the graded monad unit $\eta\colon X \to M_0 X$.

Examples

- $\mathsf{Alg}_0(\mathbb{M})$ = Eilenberg-Moore category of $(M_0, \eta, \mu^{0,0})$
- M_1**-algebra**: pair of M_0-algebras (A_0, A_1) with main structure

$$\boxed{a^{1,0} \colon M_1 A_0 \to A_1}$$

- Coherence: $(M_1 A_0, \mu_A^{0,1}) \xrightarrow{a^{1,0}} (A_1, a^{0,1})$ a homomorphism and

$$M_1 M_0 A_0 \underset{M_1 a^{0,0}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

Canonical M_1-algebras

- The **0-part** of an M_1-algebra is $(A_0, a^{0,0})$
- Taking 0-parts defines a forgetful functor

$$\mathsf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_0} \mathsf{Alg}_0(\mathbb{M}), \qquad A \mapsto (A_0, a^{0,0})$$

- An M_1-algebra A is canonical if it is free over its 0-part w.r.t. $(-)_0$.
- $\mathbb{M}$ is 'nice' $\rightsquigarrow$ $(M_0X, M_1X, \mu^{1,0})$ is canonical

Proposition [DMS19]

An M_1-algebra A is canonical iff

$$M_1M_0A_0 \underset{M_1a^{0,0}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1A_0 \xrightarrow{a^{1,0}} A_1$$

is a coequalizer diagram in $\mathsf{Alg}_0(\mathbb{M})$.

Graded equational theories

Finitary graded monads admit presentations by graded theories:

- **graded signature** Σ: algebraic signature + **depth** on operations
- **uniform-depth terms** with variables in X:

$$\frac{}{x \in T_{\Sigma,0}(X)}\ (x \in X) \qquad\qquad \frac{t_1, \ldots, t_n \in T_{\Sigma,k}(X)}{\sigma(t_1, \ldots, t_n) \in T_{\Sigma,d(\sigma)+k}}\ (\sigma \in \Sigma)$$

- **graded theory**: pair $\mathbb{T} = (\Sigma, \mathcal{E})$ with $\mathcal{E}$ a set of *u.d.* Σ-*equations*
- Sound/complete sequent-style system for graded algebraic reasoning

Theorem

Finitary graded monads are the *free-algebra monads* of a graded equational theories:

- ▷ $M_n X$ has the form $T_{\Sigma,n}(X)/{=_{\mathcal{E}}}$ for some $(\Sigma, \mathcal{E})$
- ▷ $\mathsf{Alg}_\omega(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$ (as concrete categories)

Depth-1 graded monads

- A graded theory is depth-1 if its operations/equations have depth at most 1
- $\mathbb{M}$ is depth-1 if it has a presentation by a depth-1 graded theory
  - ▷ i.e. $\mathsf{Alg}(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$ for some depth-1 graded theory $\mathbb{T}$
  - ▷ *almost* expressible in terms of a coequalizer [MPS15]
- Depth-1 graded monads are 'nice':

$$\boxed{\mathbb{M} \text{ is depth} \Rightarrow (M_k X, M_{k+1} X) \text{ is canonical.}}$$

Graded theories of trace equivalence

Graded theory of $\mathcal{A}$-traces

- Depth-0: operations/equations of join semi-lattices
- Depth-1: unary *actions* $a(-)$ satisfying $\boxed{a(x+y) = a(x) + a(y)}$

- presentation of the graded monad with $M_n X = \mathscr{P}_\omega(\mathcal{A}^n \times X)$
  - ▷ generalizes to theory of T-structured $\mathcal{A}$-traces ($M_n = T(\mathcal{A}^n \times -)$)
  - ▷ e.g. join semi-lattices $\rightsquigarrow$ convex algebras: theory of prob. traces ($T = \mathcal{D}$)
- Theories of refined trace semantics (e.g. ready, complete) obtained similarly

- Graded coalgebraic semantics -

Graded semantics

Graded semantics: framework for spectra of behavioural semantics

coalgebra [system-type] + graded monads [granularity]

Graded semantics on G-coalgebras

A pair $(\alpha, \mathbb{M})$ with $\mathbb{M}$ a graded monad and $G \xrightarrow{\alpha} M_1$ a natural transformation.

Given $X \xrightarrow{\gamma} GX$, define $\gamma^{(n)} \colon X \to M_n 1$:

$$\gamma^{(0)} := X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(n+1)} := X \xrightarrow{\alpha \cdot \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1,n}} M_{1+n} 1$$

$$x \sim_{(\alpha,\mathbb{M})} y \;:\Longleftrightarrow\; \gamma^{(n)}(x) = \gamma^{(n)}(y) \text{ for all } n \in \mathbb{N}$$

graded behavioural equivalence

Examples

Coalgebraic behavioural equivalence

Recall that $\mathbb{M}_G$ has $M_n = G^n$. Then for $(\mathsf{Id}, \mathbb{M}_G)$ we see:

- $\gamma^{(n)} \colon X \to M_n 1$ form the canonical cone into the final chain:

$$\gamma^{(0)} = X \xrightarrow{!} 1 \qquad\qquad \gamma^{(n+1)} = X \xrightarrow{\gamma} GX \xrightarrow{G\gamma^{(n)}} G^{n+1}1$$

- G finitary implies $\sim_{(\mathsf{Id}, \mathbb{M}_G)}$ captures full behavioural equivalence
- e.g. bisimilarity on LTS is captured by $G = \mathscr{P}_f(\mathcal{A} \times -)$

Trace equivalence on LTS

Let $\gamma \colon X \to \mathscr{P}_f(\mathcal{A} \times X)$ be an LTS.

▷ Trace equivalence is the relation defined for all $x, y \in X$ by

$$x \sim_{\mathsf{Tr}} y :\Longleftrightarrow \mathsf{Tr}_n(x) = \mathsf{Tr}_n(y) \text{ for all } n \in \omega$$

▷ Trace equivalence is captured by $M_n X = \mathscr{P}_f(\mathcal{A}^n \times X)$ and $\alpha = \mathsf{id}$.

- (Pre-)determinization -

(Pre-)determinization

Assumption: $(\alpha, \mathbb{M})$ a depth-1 graded semantics on G-coalgebras

- Each M_0-algebra $(A_0, a^{0,0})$ extends to a canonical algebra EA:

$$M_1 M_0 A_0 \underset{M_1 a^{0,0}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

- This assignment is part of a functor $E\colon \mathsf{Alg}_0(\mathbb{M}) \to \mathsf{Alg}_1(\mathbb{M})$
- Define

$$\boxed{\overline{M_1} := \mathsf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathsf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathsf{Alg}_0(\mathbb{M})}$$

- e.g. $\overline{M_1}(M_0 X, \mu^{0,0}) = (M_1 X, \mu^{0,1})$

- Where $F \dashv U \colon \mathsf{Alg}_0(\mathbb{M}) \to \mathsf{Set}$ is the canonical adjunction:

$$\overline{M_1}(M_0X, \mu^{0,0}) = (M_1X, \mu^{0,1}) \quad \Longrightarrow \quad U\overline{M_1}F = M_1$$

- Given $\gamma \colon X \to M_1X = U\overline{M_1}FX$, transposition yields

$$\boxed{\gamma^{\#} \colon FX \to \overline{M_1}FX}$$

- Explicitly, this map is given by Kleisli extension:

$$\gamma^{\#} = M_0X \xrightarrow{M_0\alpha\cdot\gamma} M_0M_1X \xrightarrow{\mu^{0,1}} M_1X = (\alpha\cdot\gamma)_0^*$$

Determinization

$M_01 = 1 \Longrightarrow x \sim_{(\alpha,\mathbb{M})} y$ iff $\eta(x), \eta(y)$ are finite-depth $\overline{M_1}$-behaviourally equivalent.

- Behavioural equivalence games -

The syntactic equivalence game $\mathcal{G}_n$

$\mathcal{G}_n$ captures $(\alpha, \mathbb{M})$-equivalence at depth n on

$$\gamma\colon X \to GX \qquad \rightsquigarrow \qquad \bar{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1 X)$$

Position	Player	Admissible Moves
$(s,t) \in (M_0X)^2$	D	$\{Z \subseteq (M_0X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0X)^2$	S	$Z = \{(s,t) \in (M_0X)^2 \mid (s,t) \in Z\}$

$$\text{Match of } \mathcal{G}_n : \quad (s,t)\ Z_1\ (s_1,t_1)\ \dots\ Z_n\ \boxed{(s_n,t_n)}$$

$$\boxed{\mathbb{T}_{\mathbb{M}} \vdash_0 s_n\tau = t_n\tau \quad (\tau\colon X \to 1)}$$

Slogan: equivalence games play out equational proofs in graded theories

Main theorem

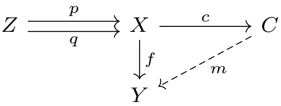
Theorem

Suppose $(\alpha, \mathbb{M})$ is depth-1 and $\overline{M_1}$ preserves monos. Then:

$$x \sim_{(\alpha,\mathbb{M})} y \iff \text{D wins } \mathcal{G}_n \text{ for all } n \in \omega$$

Currently, restricted to graded semantics in Set:

- We use that the EM category of a monad on Set is regular...
- ...ensuring that for the kernel pair $p, q\colon Z \to X$ of a $f\colon X \to Y$ we have m monic:

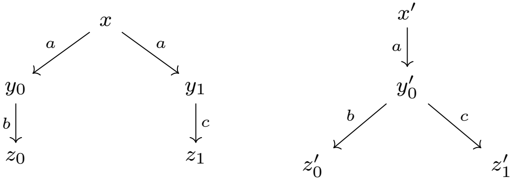


Bisimilarity game

Position	Player	Admissible Moves
$(s,t) \in (M_0 X)^2$	D	$\{Z \subseteq (M_0 X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0 X)^2$	S	$Z = \{(s,t) \in (M_0 X)^2 \mid (s,t) \in Z\}$

- $(\mathsf{id}, \mathbb{M}_{\mathscr{P}_f(\mathcal{A} \times -)})$ captures bisimilarity on f.b. LTS
- Positions for D are state pairs in a LTS $\gamma\colon X \to GX$ since $M_0 = \mathsf{Id}$
- Z is admissible for D at (x, y) if it is a *local bisimulation* at (x, y)
- Given Z, S picks the next state pair to continue the game
- D wins every full play because the $M_0 1 = 1$

Trace equivalence game



- At (x, x'), D plays $Z_1 := \{y_0 + y_1 = y'_0\}$

  admissible: $Z_1 \vdash_1 a(y_0) + a(y_1) = a(y'_0)$?
- At position Z_1, S must play $(y_0 + y_1 = y'_0) \in Z_1$
- At $(y_0 + y_1, y)$, D plays $Z_2 := \{z_0 = z'_0, z_1 = z'_1\}$

  admissibile: $Z_2 \vdash_1 b(z_0) + c(z_1) = b(z'_0) + c(z'_1)$?
- S plays a challenge from Z_2 inducing $\boxed{(x, x')\ Z_1\ (y_0 + y_1, y'_0)\ Z_2\ (z_i, z'_i)}$

  D wins: $* = *$ is valid in JSL

Concluding remarks

`See here for our arXiv preprint:`

- Graded semantics: unifying framework for linear-time/branching-time style spectra
- In this talk:
  - ▷ a generic determinization construction under graded semantics
  - ▷ game characterizations of graded behavioural equivalences *for free*
- Many interesting problems for future work:
  - ▷ fixpoint theory of graded semantics
  - ▷ extensions beyond Set (e.g. games for preorders, metrics, etc.)
  - ▷ generic minimization/L^*-style learning algorithms

References

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